Filter Representations of Normal Lattices

El-Bachir Yallaoui

Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O.Box 36, Al-Khod 123, Muscat, Sultanate of Oman.

ABSTRACT: In this paper we will investigate the properties of normality lattices and their relationships to zero-one measures. We will establish necessary and sufficient conditions for lattices to be normal. These properties are then investigated in the case of separated lattices.

Let \( X \) be an arbitrary set and \( \mathcal{L} \) a lattice of subsets of \( X \). Let \( A(\mathcal{L}) \) be the algebra generated by \( \mathcal{L} \), and let \( I(\mathcal{L}) \) be the set of non-trivial zero-one valued finitely additive measures on \( A(\mathcal{L}) \). \( I(\mathcal{L}) \) will denote those \( \mu \in I(\mathcal{L}) \) that are \( \mathcal{L} \)-regular, and \( I^*(\mathcal{L}) \) consists of those \( \mu \in I(\mathcal{L}) \) which are countably additive.

We will consider a number of equivalent characterizations of \( \mathcal{L} \) being a normal lattice. We then associate with \( \mathcal{L} \), a lattice \( W_p(\mathcal{L}) \) in \( I(\mathcal{L}) \) such that \( I(\mathcal{L}) \) is the set of non-trivial zero-one valued finitely additive measures on \( A(\mathcal{L}) \). Assuming \( A \) is countably additive, \( W_p(\mathcal{L}) \) is always a replete lattice. We will give necessary and sufficient conditions for \( W_p(\mathcal{L}) \) to be a prime complete lattice. Next, we consider the set \( I^*(\mathcal{L}) \) with the topology of closed sets given by \( \tau_{W_p(\mathcal{L})} \) consisting of arbitrary intersections of sets of \( W_p(\mathcal{L}) \). We investigate this topological space on some extent giving necessary and sufficient conditions for it to be \( T_1 \) - Lindelöf and normal:

The notations and terminology used in this paper are standard and are consistent with (Alexandroff 1937), (Bachman et al. 1983), (Szeto 1979), (Wallman 1938) and Yallaoui (1991). Our work on normal lattices is closely related to work done in Camacho (1991). We begin with a brief review of some notations and some definitions for the reader's convenience.

**Definitions and Notations**

Let \( X \) be an abstract set and let \( \mathcal{L} \) be the lattice of subsets of \( X \). We will always assume that \( 0 \) and \( X \) are in \( \mathcal{L} \). If \( A \) is a subset of \( X \) then we will denote the complement of \( A \) by \( A' \) i.e. \( A' = X - A \) and if \( \mathcal{L} \) is a lattice of subset of \( X \) then \( \mathcal{L}' \) is defined as \( \mathcal{L}' = \{ L' | L \in \mathcal{L} \} \).

**Lattice Terminology**

**Definition 1:** Let \( \mathcal{L} \) be a lattice of subsets of \( X \) be a lattice of subsets of \( X \). We say that \( \mathcal{L} \) is:

1. \( \delta \)-lattice if it is closed under countable intersections,
2. separating or \( T_1 \) if for \( x, y \in X \) and \( x \neq y \) then there exists \( L \in \mathcal{L} \) such that \( x \in L \) and \( y \notin L \),
3. Hausdorff or \( T_2 \) if for \( x, y \in X \) and \( x \neq y \) then there exists \( A, B \in \mathcal{L} \) such that \( x \in A, y \in B \) and \( A \cap B' = 0 \),
4. disjunctive for \( x \in X \) and \( L \in \mathcal{L} \) where \( x \notin L \), \( \exists A \in \mathcal{L} \) such that \( x \in A \) and \( A \cap L = 0 \),
5. normal for \( A, B \in \mathcal{L} \) where \( A \cap B = 0 \), \( \exists A', B' \in \mathcal{L} \) such that \( A \in A', B \in B' \) and \( A' \cap B' = 0 \),
6. compact if any covering of \( X \) by \( \mathcal{L} \)-sets has a finite subcovering,
7. countably compact if any countable covering of \( X \) by \( \mathcal{L} \)-sets has a finite subcovering,
8. \( \mathcal{L} \) is Lindelöf if any covering of \( X \) by \( \mathcal{L} \)-sets has a countable subcovering.

We let:

- \( A(\mathcal{L}) \) = the algebra generated by \( \mathcal{L} \),
- \( \sigma(\mathcal{L}) \) = the \( \sigma \) algebra generated by \( \mathcal{L} \),
- \( b(\mathcal{L}) \) = the lattice of countable intersections of sets of \( \mathcal{L} \),
- \( \tau(\mathcal{L}) \) = the lattice of arbitrary intersections of sets of \( \mathcal{L} \).

**Measure Terminology**

Let \( \mathcal{L} \) be a lattice of subsets of \( X \), then \( M(\mathcal{L}) \) will denote the set of finite valued bounded finitely additive measures on \( A(\mathcal{L}) \). Clearly, since any measure in \( M(\mathcal{L}) \) can be written as a difference of two non-negative measures there is no loss of generality in assuming that the
measures are non-negative, and we will assume so throughout this paper. We will say that a measure $\mu$ of $M(\mathcal{L})$ is regular if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup_{L \leq A, L \in \mathcal{L}} \mu(L)$. $M_0(\mathcal{L})$ represents the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$.

**Definition 2**:  
1. A measure $\mu \in M(\mathcal{L})$ is said to be $\sigma$-smooth on $\mathcal{L}$, if for $L_x \in \mathcal{L}$ and $L_x \uparrow \emptyset$, then $\mu(L_x) \rightarrow 0$.
2. A measure $\mu \in M(\mathcal{L})$ is said to be $\sigma$-smooth on $\mathcal{A}(\mathcal{L})$, if for $A_x \in \mathcal{A}(\mathcal{L})$ and $A_x \uparrow \emptyset$, then $\mu(A_x) \rightarrow 0$.

If $\mathcal{L}$ is a lattice of subsets of $X$, then $M(\mathcal{L}) = \mathcal{A}(\mathcal{L})$, $M(\mathcal{L})_0 = \mathcal{A}(\mathcal{L})_0$, and $M^0(\mathcal{L})$.

**Definition 3**:  
If $A \in \mathcal{A}(\mathcal{L})$ and if $x \in X$ then 
$$\mu_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases}$$
is the measure concentrated at $x$.

If $\mathcal{I}(\mathcal{L})$ is the subset of $M(\mathcal{L})$ which consist of non-trivial zero-one valued measures, then

- $I_0(\mathcal{L})$ is the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$.
- $I_0(\mathcal{L})$ is the set of $\sigma$-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$.
- $I(\mathcal{L})$ is the set of $\sigma$-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$.
- $I_0(\mathcal{L})$ is the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$.

**Definition 4**: If $\mu \in M(\mathcal{L})$, then we define the support of $\mu$ to be $\text{Supp}(\mu) = \cap \{ L \in \mathcal{L} : \mu(L) = \mu(X) \}$.  
Consequently if $\mu \in I(\mathcal{L})$, then $\text{Supp}(\mu) = \cap \{ L \in \mathcal{L} : \mu(L) = 1 \}$.

**Definition 5**: We say that the lattice $\mathcal{L}$ is:

1. Replete if $S(\mu) \neq \emptyset$ for any $\mu \in I_0(\mathcal{L})$.
2. Prime Complete if $S(\mu) \neq \emptyset$ for any $\mu \in I_0(\mathcal{L})$.

We now list a few well known facts which will enable us to characterize some previously defined properties in a measure theoretic fashion. The lattice $\mathcal{L}$ is:

1. disjunctive if and only if $\mu_x \in I_0(\mathcal{L}) \forall x \in X$,
2. $\mathcal{T}_2$ if and only if $S(\mu) = \emptyset$ or a singleton for any $\mu \in I(\mathcal{L})$,
3. compact if and only if $S(\mu) \neq \emptyset$ for any $\mu \in I(\mathcal{L})$,
4. countably compact if and only if $I_0(\mathcal{L}) = I^c_0(\mathcal{L})$,
5. Lindelöf if and only if $S(\mu) \neq \emptyset$ for any $\mu \in I_0(\mathcal{L})$,
6. normal if and only if for any $\mu \in I(\mathcal{L})$ there exists a unique $v \in I_0(\mathcal{L})$ such that $\mu \leq v$ on $\mathcal{L}$.

**Filter and Measure Relationships**

Let $\mathcal{F}$ be a lattice of subsets of $X$.

**Definition 6**: We say that $\mathcal{F}$ is an $\mathcal{L}$-filter if:

1. $\emptyset \notin \mathcal{F}$
2. If $L_1, L_2 \in \mathcal{F}$ then $L_1 \cap L_2 \in \mathcal{F}$
3. If $L_1 \subseteq L_2$ and $L_1 \subseteq \mathcal{F}$ then $L_2 \subseteq \mathcal{F}$

**Definition 7**: $\mathcal{F}$ is said to be a prime $\mathcal{L}$-filter if:

1. $\emptyset$ is an $\mathcal{L}$-filter, and
2. If $L_1, L_2 \in \mathcal{L}$ and $L_1 \cup L_2 \in \mathcal{L}$ then $L_1 \in \mathcal{L}$ or $L_2 \in \mathcal{L}$

**Definition 8**: If $\mathcal{F}$ is an $\mathcal{L}$-filter we say that $\mathcal{F}$ is an $\mathcal{L}$-ultrafilter if $\mathcal{F}$ is a maximal $\mathcal{L}$-filter.

If $\mu \in I(\mathcal{L})$ let $\mathcal{F}_\mu = \{ L \in \mathcal{L} : \mu(L) > 0 \}$.

**Proposition 9**:  
1. If $\mu \in I(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-prime filter and conversely any $\mathcal{L}$-prime filter determines an element $\mu \in I(\mathcal{L})$ and the correspondence is a bijection.
2. If $\mu \in I_0(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter and conversely any $\mathcal{L}$-ultrafilter determines an element $\mu \in I_0(\mathcal{L})$. This correspondence is also a bijection, $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter if and only if $\mu \in I_0(\mathcal{L})$.

**Separation of Lattices**

We are going to state a few known facts about the separation of lattices. We will use these results later on in the paper.

**Definition 10**: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of $X$. We say that $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for any $A_2, B_2 \subseteq \mathcal{L}_2$ and $A_2 \cap B_2 = \emptyset$ then there exists $A_1, B_1 \subseteq \mathcal{L}_1$ such that $A_2 \subseteq A_1$, $B_2 \subseteq B_1$, and $A_1 \cap B_1 = \emptyset$.

**Proposition 11**: Let $\mathcal{L}_1$ be a lattice of subset of $X$. Then $\mathcal{L}_1$ is compact if and only if $\tau \mathcal{L}_1$ is compact, in which case $\mathcal{L}_1$ separates $\tau \mathcal{L}_1$.

**Proposition 12**: $\mathcal{L}_1$ is Lindelöf if and only if $\tau \mathcal{L}_1$ is Lindelöf and if we further assume that $\mathcal{L}_2$ is also $\delta$ then $\mathcal{L}_1$ separates $\tau \mathcal{L}_2$.

The proofs for these propositions are easy and will be omitted.

**Theorem 13**: Suppose $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$. Then $\mathcal{L}_1$ is normal if and only if $\mathcal{L}_1$ is normal.
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Proof.
1. Suppose that \( A_i \) is normal and let \( A_2, B_2 \in \mathcal{L} \) such that \( A_2 \cap B_2 = \emptyset \). Since \( \mathcal{L} \) is normal then there exist \( A_1, B_1 \in \mathcal{L} \) such that \( A_1 \subset A_2, B_1 \subset B_2 \) and \( A_1 \cap B_1 = \emptyset \). Now since \( \mathcal{L} \) is normal there exist \( A, B \in \mathcal{L} \) such that \( A \subset A', B \subset B' \) and \( A' \cap B' = \emptyset \). Therefore \( A_2 \subset A_1 \subset A', B_2 \subset B_1 \subset B' \) and \( A' \cap B' = \emptyset \) i.e. \( \mathcal{L} \) is normal.

2. Suppose that \( \mathcal{L} \) is normal. Let \( \mu \in \Pi(\mathcal{L}) \) and assume that there exist two measures \( v_1, \tau_1 \in I_k(\mathcal{L}) \) and \( v_2, \tau_2 \in \mathcal{L} \), \( v_1 \leq \tau_1 \), \( v_2 \leq \tau_2 \) on \( \mathcal{L} \). Let \( v_2, \tau_2 \) be the respective extensions of the previous measures. Note that the latter two extensions are unique and belong to \( I_k(\mathcal{L}) \). Furthermore it can be seen that \( \mathcal{L} \) separates \( \mathcal{L} \) then \( \mu \leq v_2 \) and \( \mu \leq \tau_2 \). However since \( \mathcal{L} \) is normal then \( v_2 \leq \tau_2 \). Therefore \( v_1 \leq \tau_1 \) and hence \( \mathcal{L} \) is normal.

The Wallman Space

If \( \mathcal{L} \) is a disjunctive lattice of subsets of an abstract set \( X \) then there is a Wallman space associated with it. We will briefly review the fundamental properties of this Wallman space.

For any \( A \in \mathcal{A}(\mathcal{L}) \), define \( W(A) \) to be \( W(A) = \{ \mu \in I_k(\mathcal{L}) \mid \mu(A) = 1 \} \). If \( A, B \in \mathcal{A}(\mathcal{L}) \) then:
1) \( W(A \cup B) = W(A) \cup W(B) \).
2) \( W(A \cap B) = W(A) \cap W(B) \).
3) \( W(A') = W(A)' \).
4) \( W(A) \subseteq W(B) \) if and only if \( A \subseteq B \).
5) \( W(A' \cap L) = W(B) \) if and only if \( A \subseteq B \).
6) \( W(\mathcal{L}) = L \).

Let \( W(\mathcal{L}) = \{ W(L) \mid L \in \mathcal{L} \} \).

\( W(\mathcal{L}) \) is a compact lattice, and the topological space \( I_k(\mathcal{L}) \) with closed sets \( \tau W(\mathcal{L}) \) is a compact \( T_1 \) space called the Wallman space associated with \( X \) and \( \mathcal{L} \). Since \( \mathcal{L} \) is disjunctive, it will be \( T_1 \) if and only if \( \mathcal{L} \) is normal.

In addition to each \( \mu \in M_\pi(\mathcal{L}) \) there corresponds a unique \( \hat{\mu} \in M(\tau W(\mathcal{L})) \), where \( \hat{\mu}(W(A)) = \mu(A) \) for \( A \in \mathcal{A}(\mathcal{L}) \) and conversely. Also, \( \mu \in M_\pi(\mathcal{L}) \) if and only if \( \hat{\mu} \in M_\pi(\mathcal{L}) \).

Since \( W(\mathcal{L}) \) is compact so is \( \tau W(\mathcal{L}) \), and \( W(\mathcal{L}) \) separates \( \tau W(\mathcal{L}) \) (see proposition 11). Furthermore \( \hat{\mu} \in M_\pi(\mathcal{L}) \) has a unique extension to \( \hat{\mu} \in M_\pi(\tau W(\mathcal{L})) \). Next we consider the space \( I_\pi(\mathcal{L}) \) and its topology.

DEFINITION 14: Let \( \mathcal{L} \) be a disjunctive lattice of subsets of \( X \), \( L \in \mathcal{L} \) and \( A \in \mathcal{A}(\mathcal{L}) \),
1) \( W(L) = \{ \mu \in I_\pi(\mathcal{L}) \mid \mu(L) = 1 \} \).
2) \( W(\mathcal{L}) = \{ \mu \in I_\pi(\mathcal{L}) \mid \mu(A) = 1 \} \).
3) \( W(\mathcal{L}) = \{ W(L), L \in \mathcal{L} \} = W(\mathcal{L}) \cap I_\pi(\mathcal{L}) \).

The following properties hold and are not difficult to prove.

PROPOSITION 15: Let \( \mathcal{L} \) be a disjunctive lattice. Then for \( A, B \in \mathcal{A}(\mathcal{L}) \):
1) \( W(A \cup B) = W(A) \cup W(B) \).
2) \( W(A \cap B) = W(A) \cap W(B) \).
3) \( W(A') = W(A)' \).
4) \( W(A) \subseteq W(B) \) if and only if \( A \subseteq B \).
5) \( A(W(L)) = \mu(W(L)) \).
6) \( a(W(L)) = W(L) \).

For each \( \mu \in M(\mathcal{L}) \) there corresponds a unique \( \mu' \in \Pi(W(\mathcal{L})) \), where \( \mu(A) = \mu(A) \) for \( A \in \mathcal{A}(\mathcal{L}) \) and conversely. Furthermore \( \mu \in M(\mathcal{L}) \) if and only if \( \mu \in M(\tau W(\mathcal{L})) \), and \( \mu \in M(\mathcal{L}) \) if and only if \( \mu \in M(\tau W(\mathcal{L})) \).

The lattice \( W(\mathcal{L}) \) is replete and \( \Pi(W(\mathcal{L})) \) with \( \tau W(\mathcal{L}) \) as the topology of closed sets is disjunctive and \( T_1 \). It will be \( T_1 \) if we further assume that for each \( \mu \in \Pi(\mathcal{L}) \) there exists at most one \( v \in \Pi(W(\mathcal{L})) \) such that \( \mu = v \) on \( \mathcal{L} \). A proof of the last statement can be found in Yalouns (1991).

Normal Lattices

PROPOSITION 16: \( \mathcal{L} \) is normal if and only if for each \( L \in \mathcal{L} \) where \( L \subseteq L_1 \cup L_2 \) and \( L_1, L_2 \in \mathcal{L} \), then there exists \( A_1, A_2 \in \mathcal{L} \) such that \( L_1 \subseteq L_1 \), \( L_2 \subseteq L_2 \) and \( L = A_1 \cup A_2 \).

Proof:
1. Assume that \( \mathcal{L} \) is normal and let \( L \subseteq L_1 \cup L_2 \), then \( L \cap L_1 \cap L_2 = \emptyset \) or equivalently \( (L \cap L_1) \cap (L \cap L_2) = \emptyset \).

Since \( \mathcal{L} \) is normal then there exist \( A_1, A_2 \in \mathcal{L} \) such that \( L \cap L_1 \subseteq A_1 \), \( L \cap L_2 \subseteq A_2 \) and \( A_1 \cap A_2 \). Let \( A_1 = L \cap A_1 \), and \( A_2 = L \cap A_2 \). Clearly \( A_1 \subseteq L_1 \), \( A_2 \subseteq L_2 \). Now \( A_1 \cup A_2 = \emptyset \). Let \( L_1 \cap L_2 = \emptyset \) and \( L_1, L_2 \in \mathcal{L} \), then \( A_1 \subseteq L_1 \cup L_2 \), \( A_2 \subseteq L_1 \cup L_2 \) and \( A_1 \cap A_2 = \emptyset \) and \( A_1 \cup A_2 = \emptyset \) and hence \( \mathcal{L} \) is normal.

DEFINITION 17: Let \( \pi : \mathcal{L} \rightarrow \{0,1\} : \pi \) will be called a premeasure on \( \mathcal{L} \) if \( \pi(\emptyset) = 1 \), and \( \pi \) is monotone and multiplicative i.e. \( \pi(L_1 \cup L_2) = \pi(L_1) \cdot \pi(L_2) \) for \( L_1, L_2 \in \mathcal{L} \). \( \Pi(\mathcal{L}) \) denotes all such premeasures defined on \( \mathcal{L} \). It can be easily shown that there is a one to one correspondence between elements of \( \Pi(\mathcal{L}) \) and \( \mathcal{L} \)-filters.

DEFINITION 18: Let \( \Pi(\mathcal{L}) = \{ \pi \in \Pi(\mathcal{L}) \mid \pi(L_1 \cup L_2) = \pi(L_1) \cdot \pi(L_2) \) for \( L_1, L_2 \in \mathcal{L} \}. \) Clearly \( I_\mathcal{L} \subseteq \mathcal{L} \subseteq \mathcal{L} \subseteq \Pi(\mathcal{L}) \).

Let \( \mathcal{T} = \{ L \in \mathcal{L} \mid L \cap \mathcal{A} = \emptyset \} \) for all \( A \in \mathcal{L} \) such that \( \pi(A) = \)
THEOREM 19. $\mathcal{L}$ is normal if and only if $\mathcal{I}$ is an $\mathcal{L}$-ultrafilter.

Proof:

1. Assuming that $\mathcal{I}$ is normal we have to show that:

(a) $\emptyset \notin \mathcal{I}$; which is obvious

(b) $L_1 \subseteq L_2$, $L_1 \in \mathcal{I}$ implies $L_2 \in \mathcal{I}$

(c) $L_1, L_2 \in \mathcal{I}$ implies $L_1 \cap L_2 \in \mathcal{I}$

We have to show that $L_1 \cap L_2 \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\pi(A) = 1$. Assume otherwise i.e. $L_1 \cap L_2 \cap A = \emptyset$ for some $A \in \mathcal{L}$ and $\pi(A) = 1$ where $\pi \in \Pi(\mathcal{L})$ then $(L_1 \cap L_2) \cap (L_1 \cap L_2) = 0$. Since $\mathcal{L}$ is normal, there exist $A_1, A_2 \in \mathcal{L}$ such that $L_1 \cap A \subseteq A_1', L_1 \cap A \subseteq A_2'$ and $A_1 \cap A_2 \cap A = \emptyset$. Clearly $A_1 \cup A_2 = X \Rightarrow \pi(A_1, A_2) = 1$ or $\pi(A_1) = 1$ or $\pi(A_2) = 1$. If for instance $\pi(A_1) = 1$ then $\pi(A_1 \cap A) = 1$ and $L_1 \cap A \cap \pi = 1$ which is a contradiction since $L_1 \in \mathcal{I}$.

(d) Now assume that $\mathcal{G} \in \mathcal{G}$ where $\mathcal{G}$ is an $\mathcal{L}$-ultrafilter. Assume there exists $L \in \mathcal{G}$ but $L \notin \mathcal{I}$; hence there exists $A \in \mathcal{L}$ such that $\pi(A) = 1$ but $L \cap A = \emptyset$. However since $\pi(A) = 1$ then $L \cap A = \emptyset$ for all $L \in \mathcal{G} \Rightarrow \mathcal{G} = \{ A \in \mathcal{L} : \pi(A) = \infty \}$ which is a contradiction. Therefore $\mathcal{G}$ is an $\mathcal{L}$-ultrafilter.

2. Now assume that $\mathcal{I}$ is an $\mathcal{L}$-ultrafilter. We have to show that $\mathcal{L}$ is normal i.e. if $\mu \in \Pi(\mathcal{L})$ there exists a unique $v \in I_0(\mathcal{L})$ and $v \in \mu$ on $\mathcal{L}$.

Suppose that there exist $v_1, v_2 \in I_0(\mathcal{L})$ and $\mu \subseteq v_1, \mu \subseteq v_2$ on $\mathcal{L}$.

Let $\mathcal{F}_\mu = \{ L \in \mathcal{L} : \mu(L) = 1 \}$ and $\mathcal{I}_\mu = \{ L \in \mathcal{L} : L \cap A = \emptyset \}$ for all $A \in \mathcal{L}$ such that $\mu(A) = 1$.

$\mathcal{F}_\mu$ and $\mathcal{I}_\mu$ are ultrafilters and we have $\mu \subseteq v_1 \Rightarrow \mathcal{F}_\mu \subseteq \mathcal{F}_v_1 \Rightarrow \mathcal{I}_\mu \subseteq \mathcal{I}_v_1 \Rightarrow \mathcal{I}_\mu \subseteq \mathcal{I}_v_2$ for $i = 1, 2$. Therefore $\mathcal{I}_\mu \subseteq \mathcal{I}_v_1 \subseteq \mathcal{I}_v_2$.

Furthermore we have that $\mathcal{F}_v_1 \subseteq \mathcal{I}_v_2$ and hence $\mathcal{F}_v_1 = \mathcal{F}_v_2$. Finally since all the ultrafilters are equal we get that $v_1 = v_2$ which proves that $\mathcal{L}$ is normal.

Let $\mu \in I_0(\mathcal{L})$. Define for any $E \subseteq \mathcal{L}$, $\overline{\mu}(E) = \inf_{E \subseteq E'} \mu(E')$.

Then it is easily seen that $\overline{\mu}$ is a finitely subadditive outer measure.

PROPOSITION 20: $\mathcal{L}$ is normal if and only if $K = \{ L \in \mathcal{L} : \overline{\mu}(L) = 1 \}$ is a prime $\mathcal{L}$-filter.

Proof:

Suppose $\mathcal{L}$ is normal. If $L_1, L_2 \in K$ then $\overline{\mu}(L_1) = \overline{\mu}(L_2) = 1$.

Now if $\overline{\mu}(L_1 \cap L_2) = 0$ then there exists $A \in \mathcal{L}$ such that $L_1 \cap L_2 \subseteq A$ and $\mu(A') = 0$. But then $L_1 \subseteq L_2 \cup A'$, and since $\mathcal{L}$ is normal, by proposition 16, we have $A = A_1 \cup A_2$ where $A_1, A_2 \in \mathcal{L}$, $A_1 \subseteq L_1'$, and $A_2 \subseteq L_2'$ Now $\mu(A_1) = 1$ then $\mu(A_2) = 1$ or $\mu(A_1) = 1$. If for instance $\mu(A_1) = 1$, then $\mu(A_1') = 0$ which is a contradiction since $L_1 \subseteq A_1'$ and $\overline{\mu}(L_1) = 1$.

Thus $L_1 \subseteq A_1$ implies $L_1 \cap L_2 \subseteq L_2 \subseteq K$.

The converse of the proof is easy and left to reader.

THEOREM 21. Let $\pi \in \Pi(\mathcal{L})$. Then $\pi \in \Pi(\mathcal{L})$ if and only if there exists $\mu \in I(\mathcal{L})$ such that $\mu \leq \pi$.

Proof:

Suppose $\pi \in \Pi(\mathcal{L})$ and let $\mathcal{N} = \{ L' \in \mathcal{L} : \pi(L') = 0 \}$ $\emptyset \notin \mathcal{N}$ and $\mathcal{N}$ has the finite intersection property. The intersection of elements of $\mathcal{N}$ form an $\mathcal{L}$-filter base. Now assume that $\mathcal{N} = G$ and $G$ is an $\mathcal{L}$-ultrafilter. Then $G \ni \mu \in I_0(\mathcal{L})$ and therefore $\mu \in I(\mathcal{L})$, and if $\pi(L) = 0$ then $L' \in \mathcal{N} \cap G \Rightarrow \mu(L') = 0$. Hence $\mu \leq \pi$ on $\mathcal{L}$ and therefore $\exists v \in \mathcal{N}(\mathcal{L})$ such that $\mu = \pi \leq \mu$ on $\mathcal{L}$.

The second part of the proof is easy and is omitted.

Let $\mathcal{W} \subseteq \mathcal{L}$ and if $L_1, L_2 \in \mathcal{W}$ then $L_1 \subseteq L_2 \subseteq \mathcal{W}$. Consider the set of all $\mathcal{L}$-filters Ga that exclude $\mathcal{W}$, (i.e. $Ga \cap \mathcal{W} = \emptyset$). We partially order $\mathcal{G}$ by set inclusion. Since $\{X\}$ is an $\mathcal{L}$-filter that exclude $\mathcal{W}$, then there exists at least one $Ga$. Furthermore, since $\{Ga, \varepsilon\}$ is a partial ordering, which is an inductive ordering then by Zorn's lemma there must exist a maximal element. Let $G$ be this maximal element so that $G = \text{max} \{ Ga : Ga \text{ are } \mathcal{L}-\text{filters that exclude } \mathcal{W} \}$ and $G \neq \emptyset$.

THEOREM 22. $G$ is a prime $\mathcal{L}$-filter.

Proof:

$G$ is certainly an $\mathcal{L}$-filter.

Let $A \subseteq B \subseteq G$ where $A, B \subseteq \mathcal{L}$. We have to show that $A \subseteq G$ or $B \subseteq G$. Assume otherwise; that is $A, B \notin G$. Suppose that $\exists F_0 \in G$ such that $A \cap F_0 = \emptyset$ then $F_0 \cap (A \cup B) \subseteq G \Rightarrow (F_0 \cap A) \cup (F_0 \cap B) = F_0 \cap B \subseteq G \Rightarrow B \subseteq G$ which is a contradiction and we may now assume that $A \cap F = \emptyset$ and $B \cap F = \emptyset$ for all $F \subseteq G$.

Let $\mathcal{I}_A$ be the filter generated by all $\{ A \cap F \mid F \in G \}$. Since $A \subseteq \mathcal{I}_A$ and $A \notin G \Rightarrow G \subseteq A$, similarly let $\mathcal{I}_B$ be the filter generated by all $\{ B \cap F \mid F \in G \}$, $G \subseteq \mathcal{I}_B$. So there exists $H_1 \in \mathcal{I}_A, H_2 \in \mathcal{I}_B$ such that $A \cap F_0 \cap H_1$ for some $F_0 \subseteq G$. Similarly there exists $H_2 \in \mathcal{I}_A, H_2 \in \mathcal{I}_B$ such that $A \cap F_0 \cap H_2$ for some $F_0 \subseteq G$.

Let $F_1 \cap F_2 = F_3$ then $F_1 \cup F_2 = (A \cap F_1) \cup (B \cap F_2) = (A \cap F_3)$.
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\( \cap F_1 \cup (B \cap F_2) = H_1 \cup H_2 \supset (A \cup B) \cap F_3 \in G. \)
However since \( H_1 \cup H_2 \in \mathcal{H}, \) we have a contradiction. Thus \( A \in G \) or \( B \in G \) or equivalently \( G \) is a prime filter and so \( G = \mu \in \mathcal{IR}(\mathcal{L}). \)

COROLLARY 23: Let \( \pi \in \Pi(\mathcal{L}) \) then \( \pi = \bigwedge_{\mu \in \mathcal{L}} \mathcal{L}. \)

Proof:
Let \( \mathcal{F} \) be the \( \mathcal{L} \)-filter representing \( \pi \) i.e. \( \mathcal{F} = \{ L \in \mathcal{L} : \pi(L) = 1 \} \). Let \( G_\mu \) be the prime \( \mathcal{L} \)-filter representing \( \mu \), so that \( G_\mu = \{ L \in \mathcal{L} : \mu(L) = 1 \} \).

Clearly \( \mathcal{F} \subset \bigcap_{\mu \in \mathcal{L}} G_\mu \). We have to show that \( \mathcal{F} = \bigcap_{\mu \in \mathcal{L}} G_\mu \).

Assume that there exists \( A \in \mathcal{L} \) and \( A \notin \bigcap_{\mu \in \mathcal{L}} G_\mu \) for all \( \mu \), but \( A \notin \mathcal{F} \). Let \( \mathcal{F} = A \) then \( \mathcal{F} \notin A \) i.e. \( A \notin \mathcal{F} \).

Let \( \mathcal{G} \) be a maximal \( \mathcal{L} \)-filter containing \( \mathcal{F} \) and excluding \( \mathcal{F} \).

From the previous theorem, \( \mathcal{G} \) is a prime \( \mathcal{L} \)-filter and \( A \notin \mathcal{G} \), which is a contradiction; since \( A \) belongs to all prime \( \mathcal{L} \)-filters that contain \( \mathcal{F} \). Therefore \( \mathcal{F} = \bigcap_{\mu \in \mathcal{L}} G_\mu \) and hence \( \pi \in \Pi(\mathcal{L}) \).

\( \mathcal{L} \). Thus \( \pi = \bigwedge_{\mu \in \mathcal{L}} \mu \).

References


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