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# Using Switching Linear Systems to Reach a Prespecified Target

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## استعمال النظم الخطية التبديلية للوصول إلى نهاية معلومة

علي بن مرزوقة

خلاصة : في هذه الدراسة نعرض طريقتين لحل التبديل بين نظم خطية باستعمال اقتران مراقبة محدود ، والهدف الأساسي من هذه الدراسة هو إيجاد سلسلة المراقبة أو التحكم $\{U^i(K), i = 1, ..., M \text{ and } K = 0, 1, ..., N-1\}$ التي تنقل النظام من بداية معلومة إلى نهاية معلومة (أو قريبة منها) باستعمال الطريقتين : الطريقة التعدادية والطريقة الجديدة . والاستعمال المكثف للحاسب الآلي للنظام المأخوذ من تجزئة النظام المتواصل وجد أن الطريقة الجديدة أكثر فاعلية من الطريقة التعدادية ، سواء من ناحية الحسابات أو التخزين ، ويمكن الحصول على النتائج ذاتها من خلال قيم مختلفة لسلسلة المراقبة . ويمكن تعميم الطريقة الجديدة لحل أنواع مختلفة من النظم الخطية التبديلية . ولحل المسألة استخدم اقتران ذا كلفة مربعة للنظم ذات العوامل الثابتة بالنسبة للزمن .

ABSTRACT: A conventional enumeration approach, and a new approach for the solution of the control of switched linear systems with input constraints are presented. The main objective in this work is to determine control sequences $\{\mathbf{U^i(k)}, i = 1, ..., M \text{ and } k = 0, 1, ..., N-1\}$ which transfer the system from a given initial state X(0) to a specific target state $X_T$ (or to be as close as possible). Considering both the conventional enumeration and the new approach, extensive computer simulations are performed using the discrete time system obtained by sampling (or discretizing) a continuous system. The new approach is found to be more efficient than the enumeration one in terms of computations and computer storage, and performs adequately under a variety of input data. The procedure developed can be generalized and used to solve several versions of the switching control problem. In particular, a procedure using a quadratic cost function (distance) is given for problems with time-invariant coefficients.

The controllability properties of dynamical systems are of major importance in control theory, namely, finding the optimal control. The controllability properties of dynamical systems are of major importance in control theory, namely, finding the optimal control U(t). The work presented in this study focuses on the control of a certain class of dynamical systems, that is, the switching systems. The work that has been done to date is primarily in the area of control and analysis of continuous bilinear systems that have the following form:

$$\frac{dX(t)}{dt} = \sum_{i=1}^{M} U^i(t).A_i.X(t) \qquad (1)$$

where $X(t) \in \mathbf{R}^n$, $A_i$ are n-by-n matrices, and $U^i : [0, \infty) \to U(t)$ where $U(t) \in \mathbf{R}$.

The major focus of attention to date has been on the structural aspects of bilinear systems as shown by Aslanis (1983) where $X(t) \in \mathbf{R}^2$ and $U(t) = \{0, 1\}$. Tarn *et al* (1973) tested the controllability of discrete bilinear systems. Goka *et al* (1973) gave necessary and sufficient conditions for the controllability of a class of discrete bilinear systems. All the work done in discrete bilinear systems, considers the control U(k) as a scalar belonging to a compact set U such that $U(k) \leq \delta$. The case where U(k) takes just two values as an ON-OFF system (i.e., U(k) = 0 or 1) as shown by Benmerzouga (1985), has had limited treatment. An attempt to solve such a problem is done in this paper. The problem of major interest is determining control sequences $\{\mathbf{U^i(k)}, i = 1, ..., M \text{ and } k = 0, 1, ..., N-1\}$ which transfer the system from a given initial state X(0) to a specific target state $X_T$ (or to be as close as possible). A conventional enumeration approach and a new approach are treated in this work. They use the properties of a discrete time system derived by sampling (or discretizing) a continuous system. Two major advantages of the new approach are observed. The first one is a reduction in computation and storage requirements compared to the conventional enumeration approach. The second advantage is the easy way to find the control sequences $\{\mathbf{U^i(k)}, i = 1, ..., M \text{ and } k = 0, 1, ..., N-1\}$ that give the best performance. Problem definition and how to obtain the discrete system from the continuous system are given in Section 3. The performance index used is described in Section 4. In Section 5, all details related to the new approach are given. Some concluding remarks are made in Section 6.

### Problem Definition

The switching control of dynamical systems can be studied for both continuous and discrete time systems. The emphasis, in this paper, is mainly about discrete time systems obtained by sampling or discretizing a continuous system. This choice is made because of the following main two reasons: (i) most control algorithms are implemented on a digital computer, and (ii) when discretizing a continuous system the discrete system obtained will have a non-singular characteristic matrix, see Wismer and Chattergy (1978). Such ground will facilitate the computations of the switching controls, as will be shown in the subsequent sections.

### Dynamics of the Continuous System

The switching control system is interpreted as follows: the control "$U^i(t)$" can have just two values either 0 or 1. The dynamics of the continuous switching system is given by equation (1) and the following constraints:

$$U^i(t).U^j(t)=0,\forall i\neq j, \quad \text{and} \quad \sum_{i=1}^{M} U^i(t) = 1. \tag{2}$$

where M is the number of systems that can be used when steering the system from the initial state X(0) to the target state $X_T$.

The added constraints will give a very interesting dynamics of the system. Since just one control U(t) is used at each switch, the dynamics will be left with only one system, i.e., if $U^i(t) = 1$, then just the system with state matrix $A_i$ will be active. Therefore at any time, there will be exactly one system in use. Hence, at any time, the dynamics is ON by just a single system with its corresponding state matrix $A_i$. Therefore, the only task left is to find a solution to the dynamics given by equations (1) and (2) together.

It is well known in the literature of control theory of linear systems, see Brockett (1970), Kailath (1980) and Sandell and Athans (1974), that the solution of a dynamical system with only one system active, in the interval [0, t] is given by:

$$X(t) = e^{Bt}.X(0), \qquad (X(0)\ is\ given). \tag{3}$$

where the matrix B is equal to any of the given state matrices $A_i$'s, depending on which control $U^i(t)$ is equal to 1, i.e., which system is actually active.

The transition matrix $e^{Bt}$ can be computed by any standard method. Of course, each of these methods has its advantages and disadvantages as described in Hirsch and Smale (1974), Hornbeck (1975), Ralston (1965), Strang (1976) and Wilkinson (1965). The method used in this work is the eigenvalue-eigenvector method. Hence the transition matrix is given by:

$$e^{Bt} = Te^{(TBT^{-1})t}T^{-1} \tag{4}$$

where T is the eigenvector matrix, and $TBT^{-1}$ is a diagonal matrix. Given the fact that diagonalizing a matrix can be obtained even when the eigenvalues are not distinct. All computations will be carried out for systems with distinct eigenvalues.

### Dynamics of the Discrete System

As stated before, the way the discrete system is obtained is just by sampling or discretizing the solution of the continuous system given by (3). Since the initial state X(0) is given, then the dynamics of the discrete time switching system is given by the following difference equation:

$$X(k+1) = e^{B.\Delta T}.X(k), \quad \{k = 0, 1, ..., N-1\}. \tag{5}$$

Since there are M systems in the above dynamics, then the discrete time switching system is now completely defined and given by the following equation:

$$X(k+1) = \sum_{i=1}^{M} U^i(k).P_i.X(k), \tag{6}$$

where $U^i(k)$ satisfy constraint (2), $P_i = e^{A_i.\Delta T}$ {i = 1, ..., M} and ΔT is the sampling interval. The matrices $P_i$'s are obtained by discretizing the solution of the continuous system. Therefore all $P_i$'s are n-by-n nonsingular matrices as explained in the previous subsection, Freeman (1965) and Stewart (1973).

Given the dynamics (6), the main purpose of this work is to find a computationally efficient approach that will steer the system from an initial state X(0) to a target state $X_T$ (or to be as close as possible). In addition to that, to find the control sequence $\{U^i(k)\}_{k=0}^{N-1}$ that will achieve such goal in a given number of steps. Therefore, the problem is to minimize the gap (or the cost) between the final state of the system X(N) and the target state $X_T$, when steering the system from an initial state X(0) to the target state $X_T$ in a given number of steps. The gap (or cost) can be quantified as a distance, a quadratic form, or something different, depending on how we are handling the system, and what the system is actually doing. The form of the gap (or cost) and how to minimize it are described in detail in the next Section.

### The Performance Index

The performance index used is the one that minimizes the distance between the final state X(N) and the target state $X_T$ in a given number of steps, N, or the one which will give the minimum gap (or cost) when being as close as possible to the target state $X_T$. Hence the objective in this analysis is to find the sequence $\{U^i(k)\}_{k=0}^{N-1}$ that will minimize the

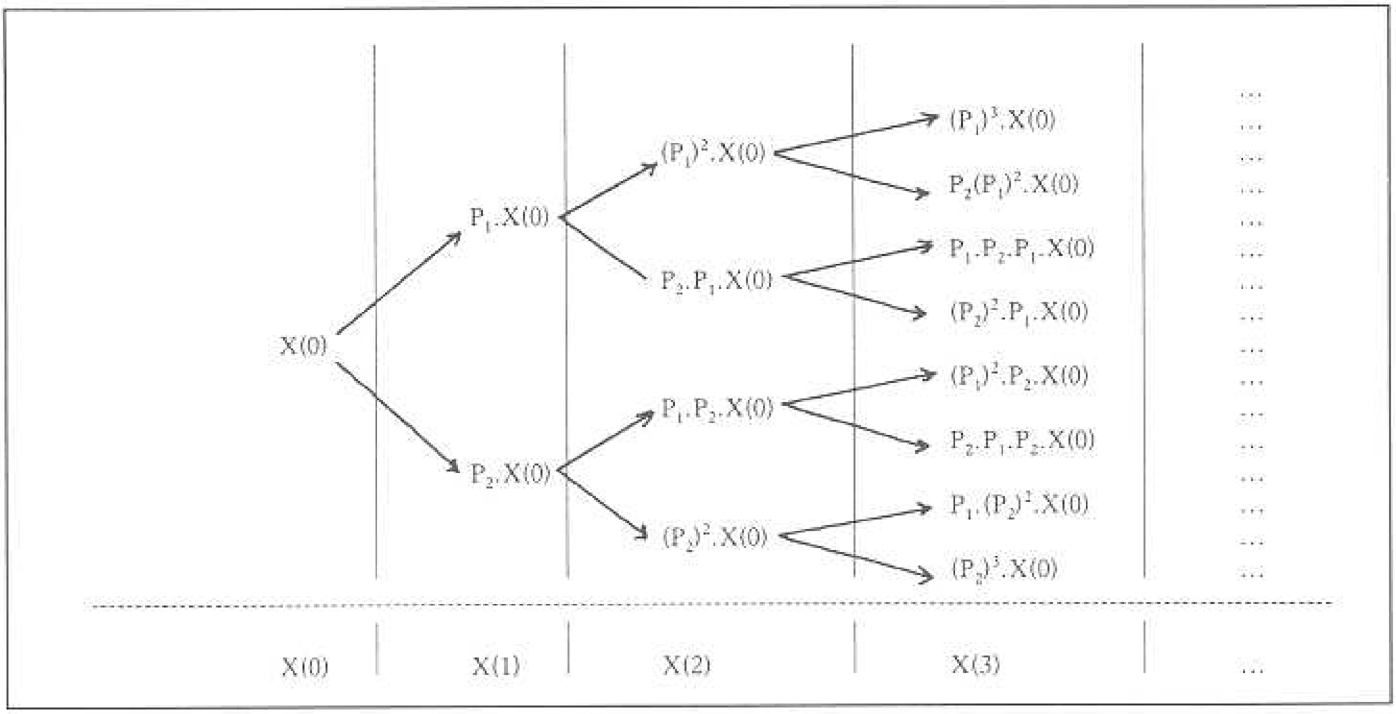


**Figure 1.** The different values of X(N) at each stage when N increases and M=2

gap (or cost) defined by the following form:

$$g[X(N)] = \|X_T - X(N)\|^2 \quad (7)$$

The criteria and objectives, mentioned in the previous section, are going to be carried out for both the conventional enumeration approach and the new approach. The following propositions will lead to the solution of the problem.

PROPOSITION 1: Given the dynamics (6), the initial state X(0), the target $X_T$ and the number of steps N, the final state X(N) can be computed as follows:

$$X(N) = \prod_{k=0}^{N-1} \sum_{i=1}^{M} U^i(k).P_i.X(0) \quad (8)$$

PROOF: Since X(0) is given, and using proof by induction,

for N = 1 $\quad X(1) = \sum_{i=1}^{M} U^i(0).P_i.X(0),$

for N = 2

$$X(2) = \sum_{i=1}^{M} U^i(1).P_i.X(1)$$
$$= \left(\sum_{i=1}^{M} U^i(1).P_i\right)\left(\sum_{i=1}^{M} U^i(0).P_i.X(0)\right)$$

hence $\quad X(2) = \left(\prod_{j=0}^{1} \sum_{i=1}^{M} U^i(j).P_i\right).X(0).$

Assuming that this is true for N = k, let us prove it for N = k + 1.
For N = k + 1

$$X(k+1) = \left(\sum_{i=1}^{M} U^i(k).P_i\right).X(k),$$

Since the statement is true for N = k, the following is obtained:

$$X(k+1) = \left(\sum_{i=1}^{M} U^i(k).P_i\right).\left(\prod_{j=0}^{k-1} \sum_{i=1}^{M} U^i(j).P_i\right).X(0),$$

and the result follows.

Without loss of generality, the analysis from now on is going to be carried out for the case of M = 2. By using the conventional enumeration method, the following three observations are made:

(i) the number of states will double from one stage to the other, as shown in Figure 1, when M = 2.
(ii) all the different states at each stage have to be computed and stored.

TABLE 1

*The Different Paths, Gaps, Combinations of $U^1(k)$, and Minimum Gaps When N = 3 and M = 2.*

| $U^1(0)$ | $U^1(1)$ | $U^1(2)$ | X(3) | $X_T$ - X(3) | g[X(3)] |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1<br>-2 | -2<br>3 | 13 |
| 1 | 1 | 0 | -1<br>1 | 0<br>0 | 0* |
| 1 | 0 | 1 | 2<br>-3 | -3<br>4 | 25 |
| 1 | 0 | 0 | -1<br>2 | 0<br>-1 | 1 |
| 0 | 1 | 1 | -1<br>1 | 0<br>0 | 0* |
| 0 | 1 | 0 | 0<br>-1 | -1<br>2 | 5 |
| 0 | 0 | 1 | -1<br>2 | 0<br>-1 | 1 |
| 0 | 0 | 0 | 1<br>-1 | -2<br>2 | 8 |

(iii) all different combinations of the control $U^i(k)$ have to be stored.

The following example will illustrate the above observations.

EXAMPLE 1: Let $P_1$ and $P_2$ be 2-by-2 matrices, X(0) and $X_T$ be 2-by-1 vectors, and the number of stages N = 3.

$$P_1 = \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix},$$

$$X(0) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \quad X_T = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

By using the conventional enumeration approach, all the states at all stages have to be computed in order to reach the last stage as shown in Figure 1. All the distances between the final state X(N) and the target state $X_T$ are given in Table 1. The minimum distances are found at row 2 and row 5 of Table 1. The solution obtained seems to be not unique. Finding the sequences of $\{U^i(k), i = 1, 2$ and $k = 0, 1, 2, 3.\}$ is not a trivial task at all. One has to go back and trace the way the optimal distances are obtained. The discussion will be carried out for the sequence $\{U^1(k), k = 0, 1, 2.\}$ only, for the second sequence $\{U^2(k), k = 0, 1, 2.\}$ can be found using constraint (2). The smallest distances when applying the conventional enumeration are found by using the sequences $\{U^1(0), U^1(1), U^1(2)\} = \{0, 1, 1\}$ and $\{U^1(0), U^1(1), U^1(2)\} = \{1, 1, 0\}$. The minimum distance between the final state X(N) and the target state $X_T$ is found to be equal to 0, which means that the system is right on target in exactly 3 steps. Of course this is not always the case. This is just a coincidence. In general, there is no guarantee to hit exactly the target in a certain number of steps.

In order to compute all different values of X(N) it is necessary to take care of all preceding values that are on that path as shown by Figure 1. The number of states at stage k+1 is the double of the one at stage k. So the number of states grows exponentially. Hence the problem gets more complicated as the number of stages increases. Therefore considerable computations and storage are required. In addition to the above, the sequence $\{U^1(k)\}_{k=0}^{N-1}$, that has a combinatorial variation, has to be determined. The following proposition will take care of such difficulties and reduces the computations and storage considerably.

PROPOSITION 2: Consider the following change of coordinates $Z(k) = B^{-k}X(k)$, where $B = P_2$ (n-by-n nonsingular matrix). The dynamics of the system in the new space is given by:

$$Z(k+1) = [I+U^1(k)\,Q(k)]Z(k) \tag{9}$$

where

$$Q(k) = B^{-k}\,[B^{-1}(P_1 - B)]B^k, \text{ and } Z(0) = X(0).$$

PROOF: Using the above change of coordinates, the dynamics given by (6) and (9), and some algebraic manipulations, Z(k+1) has the following expression:

$$Z(k+1) = Z(k) + U^1(k).B^{-k}\left[B^{-1}.(P_1 - B)\right].X(k)$$

which gives the required result.

PROPOSITION 3: Given the dynamics in (9), the matrix Q(k), has the following recurrent formula:

$$Q(k+1) = B^{-1}\,Q(k)\,B \tag{10}$$

PROOF: By using proposition 2 and mathematical induction let us see if (10) is true for all positive integer k. When k = 0, Q(0) is given by the following expression:

$$Q(0) = B^{-0}\left[B^{-1}.(P_1 - B)\right].B^0 = B^{-1}.(P_1 - B).$$

When k = 1

$$Q(1)=B^{-1}\left[B^{-1}.(P_1-B)\right].B^{1}=B^{-1}.Q(0).B.$$

Let us assume that

$Q(k)=B^{-1}.Q(k-1).B=B^{-k}.Q(0).B^{k}$ holds for k, then prove it for k + 1. Then

$$Q(k+1) = B^{-(k+1)}.\left[B^{-1}.(P_1-B)\right].B^{k+1},$$

with simple matrix manipulations the stated result follows.

PROPOSITION 4: Using propositions 2 and 3, the computations and computer storage required are reduced by half.

PROOF: By just applying the recurrent form of [I+Q(k)] on Z(0), the values of Z(N) can be computed for any stage as shown in Figure 2. For each stage, all the different values of the previous stage are found in the next stage. Therefore it is not necessary to compute all different values of Z(N), only half of them need to be computed. Furthermore, the only values of Z(N) that requires storage are those computed at stage N, all previous ones are not needed, so they are not stored.

When the different values of Z(N) are stored (or listed) as shown in Table 3, and the final state Z(N) that gives the minimum distance (or cost), is found to be at the $J^{th}$ row, finding the sequence $\{U^1(k)\}_{k=0}^{N-1}$ is no longer difficult as shown by the following proposition.

PROPOSITION 5: Suppose that the final state that gives the minimum distance (or cost) is at row J. Then the sequence $\{U^1(k)\}_{k=0}^{N-1}$ is given by the binary representation of the number (rank) J:

$$J^* = \sum_{k=0}^{N-1} a_k.2^k \tag{11}$$

where the values of $a_k$'s are either 0 or 1, and $U^1(k) = a_k$ for k = 0, 1, ..., N - 1.

PROOF: By just going back to the way Table 2 and Table 3 are constructed, the result follows.
Therefore, given any stage N, it is possible to compute all different states Z(N), starting from Z(0) and finishing with $\prod_{k=0}^{N-1}[I+Q(k)]Z(0)$. The matrix Q(k) can be computed using (10). The sequence $\{U^1(k)\}_{k=0}^{N-1}$ can be easily found by taking the binary representation of the rank (J) of the optimal state Z(N).

### The Algorithm

The algorithm is built on the basis of the previous details. The objective is to steer the system from an initial

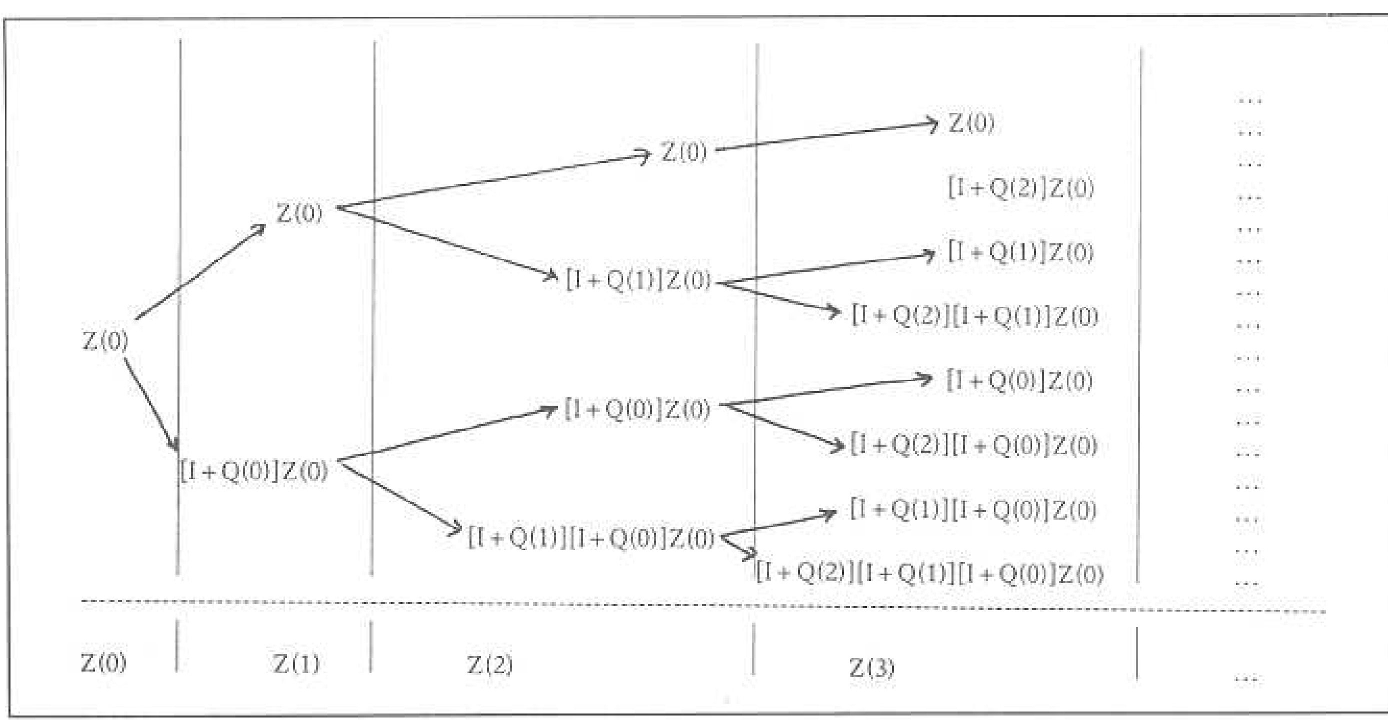


Figure 2. The different values of Z(N) at each stage when N increases.

TABLE 2

*The New way of storing the different values of Z(N) when N increases and M = 2.*

| | Z(1) | Z(2) | Z(3) | ... | Z(K) | ... | Z(N) |
|---|---|---|---|---|---|---|---|
| Z(0) | * | ○ | □ | ... | ■ | ... | ∘ |
| [I+Q(0)]Z(0) | * | ○ | □ | ... | ■ | ... | ∘ |
| [I+Q(1)]Z(0) | | ○ | □ | ... | ■ | ... | ∘ |
| [I+Q(1)][I+Q(0)]Z(0) | | ○ | □ | ... | ■ | ... | ∘ |
| [I+Q(2)]Z(0) | | | □ | ... | ■ | ... | ∘ |
| [I+Q(2)][I+Q(0)]Z(0) | | | □ | ... | ■ | ... | ∘ |
| [I+Q(2)][I+Q(1)]Z(0) | | | □ | ... | ■ | ... | ∘ |
| [I+Q(2)][I+Q(1)][I+Q(0)]Z(0) | | | □ | ... | ■ | ... | ∘ |
| ⋮ | ⋮ | ⋮ | ⋮ | ... | ⋮ | ⋮ | ⋮ |
| [I+Q(k)] [I+Q(1)][I+Q(0)]Z(0) | | | | ... | ■ | ... | ∘ |
| ⋮ | ⋮ | ⋮ | ⋮ | ... | ⋮ | ⋮ | ⋮ |
| [I+Q(N-1)] [I+Q(1)][I+Q(0)]Z(0) | | | | | | | ∘ |

state X(0) to a target state (or as close as possible) in a certain number of stages, N. Assuming that the terminal cost (or distance) is minimized, the following steps are obtained:

**Step 1:** Given the number of stages N and an initial state X(0), a matrix S(I, J) is generated which will contain all different values of Z(N) as it is shown in Table 2, where I is the dimension of the space and J the number of states at stage N.

**Step 2:** When searching for the minimum distance it is necessary to go back from Z space to the X space using the relationship $X(N) = B^N.Z(N)$. Hence a need to compute the matrix $B^N$.

**Step 3:** Compute $X(N) = B^N.Z(N)$, namely, the transformation back to X space.

**Step 4:** Compute all possible costs (or performance indices) at stage N, and keep the minimum one with its corresponding rank in the matrix S(I, J) that looks exactly like Table 3.

**Step 5:** Given the rank (J), of the minimum cost, a binary representation to get the

sequence $\{U^1(k)\}_{k=0}^{N-1}$.

The above algorithm facilitates the way to get the optimal control for the original problem. Therefore, given an initial state of the system X(0), a target state $X_T$, and a number of steps N, the algorithm will give us the optimal state and its corresponding sequence of $\{U^1(k), k = 0, 1, \ldots, N-1\}$. For the illustration of the above findings, many cases (or examples) were solved, among them, the following example. For further details see Benmerzouga (1985).

EXAMPLE 2: Let $P_1$ and $P_2$ be 2x2 matrices, X(0) and $X_T$ be 2-by-1 vectors, and the number of stages N = 3.

$$P_1 = \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix},$$

$$X(0) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \quad X_T = \begin{vmatrix} -1 \\ 1 \end{vmatrix}.$$

After computing [I + Q(k)] for k = 0, 1, and 2, the final states Z(N) are given by Table 4. The minimum distance is found to be equal to 0. The solution is found at two ranks, namely, rank 3 and 6 respectively. The binary representations of 3 and 6 will give the optimal sequences $\{U^1(0), U^1(1), U^1(2)\}=\{0, 1, 1\}$ and $\{U^1(0), U^1(1), U^1(2)\} = \{1, 1, 0\}$. Exactly the same results as Example 1 where the conventional enumeration method was used.

## Conclusions

This work presents a conventional enumeration approach and a new approach to solve a switching control problem for discrete time systems (M = 2). The discrete time system is obtained by sampling (or discretizing) the

TABLE 3

*The New way of Storing The Different Values of Z(N) When N = 3 and M = 2, their Corresponding $U^1(k)$, and Their Corresponding Rank J.*

| | Z(1) | Z(2) | Z(3) | $U^1(0)$ | $U^1(1)$ | $U^1(2)$ | Rank: J |
|---|---|---|---|---|---|---|---|
| Z(0) | • | ○ | □ | 0 | 0 | 0 | 0 |
| [I+Q(0)]Z(0) | • | ○ | □ | 1 | 0 | 0 | 1 |
| [I+Q(1)]Z(0) | | ○ | □ | 0 | 1 | 0 | 2 |
| [I+Q(1)][I+Q(0)]Z(0) | | ○ | □ | 1 | 1 | 0 | 3 |
| [I+Q(2)]Z(0) | | | □ | 0 | 0 | 1 | 4 |
| [I+Q(2)][I+Q(0)]Z(0) | | | □ | 1 | 0 | 1 | 5 |
| [I+Q(2)][I+Q(1)]Z(0) | | | □ | 0 | 1 | 1 | 6 |
| [I+Q(2)][I+Q(1)][I+Q(0)]Z(0) | | | □ | 1 | 1 | 1 | 7 |

TABLE 4

*The Different Values of Z(N) And X(N), their Corresponding Ranks, and The Minimum Gaps When N = 3 and M = 2*

| Z(3) | X(3) | g[X(3)] | Rank |
|---|---|---|---|
| 1<br>1 | 1<br>-1 | 8 | 0 |
| -2<br>-1 | -1<br>2 | 1 | 1 |
| 1<br>0 | 0<br>-1 | 5 | 2 |
| -1<br>-1 | -1<br>1 | $0^*$ | 3 |
| -2<br>-1 | -1<br>2 | 1 | 4 |
| 3<br>2 | 2<br>-3 | 25 | 5 |
| -1<br>-1 | -1<br>1 | $0^*$ | 6 |
| 2<br>1 | 1<br>-2 | 13 | 7 |

continuous system. The analysis is restricted to the case where the eigenvalues of the state matrices $A_i$ are distinct. The same results were obtained when applying both approaches to the above example. But when using the conventional enumerational approach, all the states at each stage have to be computed and stored. On the other hand just half of the states have to be computed and stored with the new approach.

There are additional good properties associated with the new approach that helped in achieving such an efficiency, i.e., the nice recurrent formula for the matrix Q(k), the back transformation from Z-space to X-space, and the matrix $B^N$. When the number of steps is small the procedure is not computationally difficult. Therefore both approaches perform basically the same. But when the number of steps increases the new approach will take over all the way. In other words, the computations and storage involved in the procedure will be cut down by at least 50%. Also, another important ingredient that improves the efficiency of the algorithm is the easy way to determine the sequence $\{U^1(k), k = 0, ..., N - 1\}$, which is cumbersome and not straightforward in the conventional enumeration approach.

The construction of the algorithm, corresponding to the new approach, was developed. The computer simulations showed that the obtained algorithm performed successfully in computing the sequences $\{U^1(k), k = 0, ..., N - 1\}$. The algorithm was shown to perform adequately when the number of steps increases.


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