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Constructing Node-Disjoint Routes in K-ary N-cubes

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إنشاء مسارات متوازية في شبكات المكعبات

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خلاصة : نعرض في هذا البحث طريقة لإنشاء مسارات متوازية داخل شبكات المكعبات . نبدأ بتقديم طريقة عامة لبناء مسارات متوازية داخل شبكات الضرب الكارتيزي ثم نطبق هذه الطريقة العامة على شبكات المكعبات التي يمكن اعتبارها ناتجة عن الضرب الكارتيزي للشبكات الكاملة . لا تتجاوز أطوال المسارات المحددة عن وحدتين زائد الأطوال المثلى. تتيح هذه المسارات المتوازية النقل السريع لكميات كبيرة من البيانات كما أنها تتيح مسارات بديلة في حالات وجود خلل في بعض نقاط المعالجة أو في الخطوط الرابطة بينها.

ABSTRACT: In this paper, a method for constructing node-disjoint (parallel) paths in k-ary n-cube interconnection networks is described. We start by showing in general how to construct parallel paths in any Cartesian product of two graphs based on known paths in the factor graphs. Then we apply the general result to build a complete set of parallel paths (i.e., as many paths as the degree of the network) between any two nodes of a k-ary n-cube which can be viewed as the Cartesian product of complete graphs. Each of the constructed paths is of length at most 2 plus the minimum distance between the two nodes. These parallel paths are useful in speeding-up the transfer of large amounts of data between two nodes and in offering alternate routes in cases of faulty nodes.

Many graphs have been studied as attractive interconnection topologies for large multiprocessor systems including the binary-hypercube (Saad and Schultz, 1988), the 2-dimensional torus (Dally and Seitz, 1986, Gravano et al, 1994), and the k-ary n-cube (Agrawal and Bhuyan, 1984, Lakshmivarahn and Dhall, 1988, Graham and Seidel, 1993). There is confusion in the literature about which graph is called the k-ary n-cube. For example, what is called the torus network in Dally and Seitz (1986) and Gravano et al (1994) is called the k-ary n-cube in Linder and Harden (1991) and in Bose et al (1995) while in Graham and Seidel (1993) the k-ary n-cube refers to a different topology. As defined later, the k-ary n-cube considered in this paper is the same as in Graham and Seidel (1993). Graham and Seidel (1993) have shown that the k-ary n-cube performs better in terms of one-to-all broadcasting and complete broadcasting than the star graph with a comparable number of nodes for practical network sizes. In general, the criteria used in evaluating interconnection networks relate to their topological properties of symmetry, scalability, low degree and diameter, efficient distributed routing algorithms, recursive structure, fault tolerance, low-cost embedding of other topologies, support of efficient broadcasting and existence of parallel paths. This paper contributes to the study of k-ary n-cubes by presenting a method for constructing complete sets of node-disjoint (parallel) paths between arbitrary nodes. These sets are complete in the sense that we obtain as many parallel paths between any two nodes as the degree of the network. Furthermore, the obtained paths are of optimal lengths plus at most 2 independently of the network size and the distance between the two nodes. Many research works have addressed such constructions of parallel paths on various interconnection networks such as the hypercube (Saad and Schultz, 1988), the star graph (Day and Tripathi, 1994), and the arrangement graph (Day and Tripathi, 1998).

Preliminaries

In the following we present a number of definitions and notations used in the paper.

DEFINITION 1: (Graham and Seidel, 1993): The k-ary n-cube Q_n^k is formed of $N = k^n$ nodes labeled by the base-k integers of the form $a_{n-1}a_{n-2}...a_0$, where $0 \le a_i < k$ for $0 \le i < n$. Two nodes are connected if and only if they differ by exactly one of their n digits.

It has been shown that Q_n^k has degree $(k-1)n$ and diameter n (Agrawal and Bhuyan, 1984).

DEFINITION 2: (Leighton, 1992): The Cartesian product $G = G_1 \otimes G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$, where V and E are given by:

i) $V = \{ <x, y> \mid x \in V_1 \text{ and } y \in V_2\}$, and
ii) If $u = <x_u, y_u>$ and $v = <x_v, y_v>$ in V then (u, v) is an edge in E iff either $(x_u, x_v) \in E_1$ and $y_u = y_v$ or $(y_u, y_v) \in E_2$ and $x_u = x_v$.

The edge (u, v) is called a G_1-edge if (x_u, x_v) is an edge in E_1, and it is called a G_2-edge if (y_u, y_v) is an edge in E_2. We call x_u the G_1-component of u and y_u the G_2-component of u.

Let N_1, δ_1, Δ_1 be respectively the size (number of nodes), degree, and diameter of G_1; and let N_2, δ_2, Δ_2 be respectively the size, degree, and diameter of G_2. The size N, degree δ, and diameter Δ of $G_1 \otimes G_2$ are given by: $N = N_1 . N_2$, $\delta = \delta_1 + \delta_2$, $\Delta = \Delta_1 + \Delta_2$. The size and degree expressions are fairly obvious. As for the diameter expression it can be justified by noticing that a path between any two vertices $u = <x_u, y_u>$ and $v = <x_v, y_v>$ of $G_1 \otimes G_2$ is composed of two types of edges: G_1-edges (affecting the G_1-component) and G_2-edges (affecting the G_2-component). If all the G_1-edges (resp. the G_2-edges) in the path from u to v are extracted and listed maintaining their relative order, we obtain a path from x_u to x_v in G_1 (resp. from y_u to y_v in G_2). Therefore u would be at a maximum distance from v in $G_1 \otimes G_2$ if, and only if, x_u is at maximum distance from x_v in G_1 and y_u is at maximum distance from y_v in G_2.

Let K_k denote the complete graph with k nodes. If we perform the Cartesian product of K_k by K_k for n times, the k-ary n-cube Q_n^k will be obtained.

LEMMA 1: Q_n^k is isomorphic to $K_k \otimes K_k \otimes ... \otimes K_k$ (n times).

Proof: This can be shown by induction on n. First, notice that Q_1^k is isomorphic to K_k (trivial). Next, we can obtain an isomorphism between Q_n^k and $K_k \otimes Q_{n-1}^k$ by mapping the node $a_{n-1}a_{n-2}...a_0$ of Q_n^k to the node $<a_{n-1}, a_{n-2}a_{n-3}...a_0>$ of $K_k \otimes Q_{n-1}^k$. By definition 2, the nodes connected to the node $a = <a_{n-1}, a_{n-2}a_{n-3}...a_0>$ in $K_k \otimes Q_{n-1}^k$ are the nodes $b = <b_{n-1}, b_{n-2}b_{n-3}...b_0>$ such that b_{n-1} is connected to a_{n-1} in K_k (i.e. $a_{n-1} \neq b_{n-1}$) or $b_{n-2}b_{n-3}...b_0$ is connected to the node $a_{n-2}a_{n-3}...a_0$ in Q_{n-1}^k (i.e. differing only in one of their n-1 positions), but not both. Therefore a and b are connected if for exactly one position i, $0 \leq i < n$, $a_i \neq b_i$ which corresponds to the same connectivity condition in Q_n^k.

Parallel Paths in a Cartesian Product

In this section we show that if G_1 and G_2 are two regular graphs such that for each one of these graphs there exists a complete family of parallel paths (i.e. as many paths as the degree of the graph) between any two of its nodes, then the same property holds for $G_1 \otimes G_2$.

We start by introducing some notations used in the construction of parallel paths. A path π in some graph from a node u_1 to a node u_m going through the intermediate nodes $u_2, u_3 ... u_{m-1}$ will be denoted by: $\pi = u_1u_2 ... u_m$. The length m-1 of such a path π is denoted by $|\pi|$. We will also denote π-1 the path from u_1 to u_{m-1} obtained from π by removing its last edge.

Consider two paths $\pi_1 = u_1u_2 ... u_m$ and $\pi_2 = v_1 ... v_t$ in some graph such that the destination node of π_1 coincides with the source node of π_2 (i.e. $u_m = v_1$). The concatenation of π_1 and π_2 denoted $\pi_1 \| \pi_2$ is the path $u_1u_2 ... u_mv_2 ... v_t$ obtained by joining the two paths π_1 and π_2.

Let x be a node in G_1 and let $\pi = y_1y_2 ... y_m$ be a path in G_2. We denote by $<x, \pi>$ the path $<x, y_1><x, y_2> ... <x, y_m>$ in $G_1 \otimes G_2$. Similarly, if $\pi = x_1 x_2 ... x_m$ is a path in G_1 and y is a node in G_2, then $<\pi, y>$ denotes the path $<x_1, y><x_2, y> ... <x_m, y>$ in $G_1 \otimes G_2$.

DEFINITION 2: A regular graph G of degree δ is said to *have complete parallel paths with maximum length increase r* if, and only if, for any two distinct nodes u and v of G, there exists a set of δ node-disjoint paths between u and v such that each of these paths is of length at most $dist(u,v)+r$, and at least one of these paths is of length $dist(u,v)$, where $dist(u,v)$ is the minimum distance between u and v in G. We use the abbreviation: G has *CPP/r-MLI*.

THEOREM 1: If G_1 is regular of degree δ_1, G_2 is regular of degree δ_2, G_1 has *CPP/r_1-MLI*, and G_2 has *CPP/r_2-MLI*, then $G_1 \otimes G_2$ has *CPP/max(2,r_1,r_2)-MLI*.

Proof: Let $u = <x_u, y_u>$ and $v = <x_v, y_v>$ be two distinct nodes in $G_1 \otimes G_2$. We distinguish two cases:

CASE 1: If $x_u \neq x_v$ and $y_u \neq y_v$.

Let $\pi_x^1, \pi_x^2, \pi_x^{\delta 1}$ be δ_1 parallel paths between x_u and x_v in G_1. These paths exist since $x_u \neq x_v$, G_1 is regular of degree δ_1 and G_1 has *CPP/r$_1$-MLI*. Each of these paths is of length at most $dist(x_u,x_v)+r_1$ and at least one of them (say π_x^1) is of length $dist(x_u, x_v)$. Similarly, there exists δ_2 parallel paths $\pi_y^1, \pi_y^2, \pi_y^{\delta 2}$ between y_u and y_v in G_2. Each of these paths is of length at most $dist(y_u, y_v)+r_2$ and at least one of them (say π_y^1) is of length $dist(y_u, y_v)$. We must therefore have:

$|\pi_x^1| = dist(x_u,x_v)$,

$2 \leq |\pi_x^i| \leq dist(x_u,x_v) + r_1$, for $2 \leq i \leq \delta_1$,

$|\pi_y^1| = dist(y_u,y_v)$,

$2 \leq |\pi_y^i| \leq dist((y_u,y_v) + r_2$, for $2 \leq i \leq \delta_2$.

Let x^i denote the last intermediate node on the path π_x^i for all i, $2 \leq i \leq \delta_1$. In other words, x^i is such that (x^i,x_v) is the last edge of π_x^i . Let also y^i denote the last intermediate node on the path π_y^i for all i, $2 \leq i \leq \delta_2$. Therefore, y^i is such that (y^i, y_v) is the last edge of π_y^i. We construct the following $\delta_1 + \delta_2$ paths between $u = \langle x_u, y_u \rangle$ and $v = \langle x_v, y_v \rangle$:

$$\pi_1 = \langle \pi_x^1, y_u \rangle || \langle x_v, \pi_y^1 \rangle,$$

$$\pi i = \langle \pi_x^i - 1, y_u \rangle || \langle x^i, \pi_y^1 \rangle || \langle x^i x_v, y_v \rangle, \ for\ all\ i, 2 \leq i \leq \delta_1,$$

$$\pi_{\delta 1+1} = \langle x_u, \pi_y^1 \rangle || \langle \pi_x^1, y_v \rangle,$$

$$\pi_{\delta 1-i} = \langle x_u, \pi_y^i - 1 \rangle || \langle \pi_x^1, y^i \rangle || \langle x_v, y^i y_v \rangle, \ for\ all\ i, 2 \leq i \leq \delta_2,$$

The path π_1 is obtained by moving along the edges of π_x^1 transforming x_u into x_v followed by the edges of π_y^1 transforming y_u into y_v Therefore π_1 is of length $dist(x_u,x_v) + dist(y_u,y_v)$ which is equal to $dist(u,v)$. The path π_i, $2 \leq i \leq \delta_1$, is obtained by following the edges of π_x^i except the last edge, followed by all the edges of π_y^1 and finally the last edge of π_x^i.

Since π_y^1 is of minimum length and π_x^i is of length at most r_1 plus the minimum length, therefore π_i is of length at most $dist(u,v)+r_1$. A similar argument is used to infer that the path $\pi_{\delta 1+1}$ is of length $dist(u,v)$ and that each path $\pi_{\delta 1+i}$, $2 \leq i \leq \delta_1$, is of length at most $dist(u,v)+r_2$.

To show that the π_i paths, $1 \leq i \leq \delta_1+\delta_2$ are node-disjoint we introduce the following notations. Let S_A be the set of all the intermediate (i.e. other than the source and the destination) nodes along all the paths π_i, $1 \leq i \leq \delta_1$, which have y_u as G_2-component. Notice that all these nodes appear first (at the left) in π_i, for each i, $1 \leq i \leq \delta_1$. Let S_B be the set of all the remaining intermediate nodes along these paths. Similarly, let S_C be the set of all the intermediate nodes along all the paths $\pi_{\delta 1+i}$, $1 \leq i \leq \delta_2$ which have x_u as G_1-component. These nodes appear first (at the left) in $\pi_{\delta 1+i}$ for each i, $1 \leq i \leq \delta_2$. Finally, let S_D be the set of the remaining intermediate nodes along these $\pi_{\delta 1-i}$ paths.

None of the sets S_A, S_B, S_C, and S_D has a node that appears in it more than once. This is justified by the fact that the π_x^i paths are node-disjoint in G_1 and the π_y^i paths are node-disjoint in G_2. The set S_A is disjoint with each of S_B, S_C, and S_D since all the nodes in S_A have y_u as G_2-component which is not the case for any node in any of the sets S_B, S_C, and S_D . Similarly, the set S_C is disjoint with each of S_B and S_D since all the nodes in S_C have x_u as G_1-component which is not the case for any node in any of the sets S_B and S_D.

It remains to show that S_B and S_D are disjoint. Let S_B^1 be the subset of nodes of S_B that appear in π_1 and let S_B^R denote the set $S_B - S_B^1$. Define S_D^1 as the subset of nodes of S_D that appear in $\pi_{\delta 1+1}$ and let S_D^R denote the set $S_D - S_D^1$. S_B^1 and S_D^1 are disjoint since all the nodes of S_B^1 have x_v as G_1-component which is not the case for any node of S_D^1. S_B^1 and S_D^R are disjoint since each node of S_B^1 has some intermediate node of π_y^1 as G_2-component which is not true for any node in S_D^R. S_B^R and S_D^1 are disjoint since each node of S_D^1 has some intermediate node of π_x^1 as G_1-component which is not true for any node in S_B^R. S_B^R and S_D^R are disjoint since each node of S_B^R has an intermediate node of π_y^1 or y_v as G_2-component which is not true for any node in S_D^R. Therefore the π_i paths, $1 \leq i \leq \delta_1 + \delta_2$, form a maximum-size family of $\delta_1+\delta_2$ node-disjoint paths between u and v and each is of length at most $dist(u,v)+\max(r_1,r_2)$. Notice that π_1 is of minimum length.

CASE 2: If $x_u \neq x_v$ and $y_u = y_v$ (the case $x_u = x_v$ and $y_u \neq y_v$ is similar)

Let $\pi_x^1, \pi_x^2, ... \pi_x^{\delta 1}$ be δ_1 parallel paths between x_u and x_v in G_1. These paths exist since $x_u \neq x_v$, G_1 is regular of degree δ_1 and G_1 has *CPP/r$_1$-MLI*. Each of these paths is of length at most $dist(x_u,x_v)+r_1$ and at least one of them (say π_x^1) is of length $dist(x_u,x_v)$. Let y_u^i, $1 \leq i \leq \delta_2$, be the δ_2 distinct adjacent nodes to y_u in G_2. We construct the following $\delta_1+\delta_2$ paths between u and v:

$\pi_i = < \pi^i_x, y_u >$, *for all i*, $1 \le i \le \delta_1$,

$\pi_{\delta 1+i} = <x_u, y_u y^i_u> \| <\pi^1_x, y^i_u> \| <x_v, y^i_u y_u>$, *for all i*, $1 \le i \le \delta_2$.

The paths π_i, $1 \le i \le \delta_1$, are node-disjoint among each other since the π^i_x, , $1 \le i \le \delta_1$, are node-disjoint in G_1. Each of these paths is of length at most $dist(u,v)+r_1$. Notice that π_1 is of minimum length. The paths $\pi_{\delta 1+i}$, for all i, $1 \le i \le \delta_2$ are node-disjoint among each other since each path has a different fixed G_2-component y^i_u at its intermediate nodes. Each $\pi_{\delta 1+i}$ path is of length $dist(u, v) + 2$. Finally, every π_i path, $1 \le i \le \delta_1$, is node-disjoint with every $\pi_{\delta 1+j}$ path, $1 \le j \le_2 \delta$, since all intermediate nodes of π_i have y_u as G -component which is not the case for any of the intermediate nodes of $\pi_{\delta 1+j}$.

Parallel Paths in the *k*-ary *n*-cube

We start by showing how to construct a complete family of parallel paths between any two nodes of a complete graph.

LEMMA 2: The complete graph on k nodes K_k has *CPP/1-MLI*.

Proof: Let 0, 1, ... , k-1 be the k nodes of K_k. Let x and y be any two distinct nodes in K_k. Hence, $0 \le x, y < k$ and $x \ne y$. Consider the following k-1 paths between x and y:

$$\pi^*_{x,y} = xy,$$

$$\pi^i_{x,y} = xiy, \text{ for all } i, 0 \le i < k, i \ne x, \text{ and } i \ne y.$$

$\pi^*_{x,y}$ is of minimum length equal to 1. Each $\pi^i_{x,y}$, $0 < i < k$, $i \ne x$, and $i \ne y$ is of length 2 and has only one intermediate node i which is different from the intermediate node j of any other $\pi^j_{x,y}$. Therefore, K_k has *CPP/1-MLI*.

The following theorem is a direct derivation from Lemma 1, Theorem 1, and Lemma 2.

THEOREM 2: The k-ary n-cube Q^k_n has *CPP/2-MLI*.

EXAMPLE: Let us follow the method described in the proof of Theorem 1 to construct step by step the parallel paths between the two nodes 0000 and 0120 of the 3-ary 4-cube Q^3_4 . By Lemma 1, Q^3_4 is isomorphic to $K_3 \otimes Q^3_3$. The node 0000 of Q^3_4 is mapped to the node <0, 000> of $K_3 \otimes Q^3_3$ and 0120 to <0, 120>. We apply the construction of Case 2 of Theorem 1.

Step 1: Find the parallel paths between 000 and 120 in Q^3_3 . By Lemma 1, Q^3_3 is isomorphic to $K_3 \otimes Q^3_2$. The node 000 is mapped to <0, 00> and 120 is mapped to <1, 20>. We need to apply the construction of Case 1 of Theorem 1.

Step 1.1: Find the parallel paths between nodes 0 and 1 in K_3. These are given by Lemma 2 as follows (we use the symbol '→' in path descriptions to denote edges in the path):

0→1
0→2→1

Step 1.2: Find the parallel paths between 00 and 20 in Q^3_2 . But, Q^3_2 is isomorphic to $K_3 \otimes K_3$ (by Lemma1). The node 00 is mapped to <0, 0> and 20 is mapped to <2, 0>. We need to apply the construction of Case 2 of Theorem 1 using the parallel paths 0→2 and 0→1→2 between nodes 0 and 2 in K_3 (by Lemma 2). This results in the following paths between 00 and 20:

00→20
00→10→20
00→01→21→20
00→02→22→20

Step 1.3: Combine the results of Step 1.1 and 1.2 using the construction of Case 1 to obtain the following paths between 000 and 120:

000→100→120
000→200→220→120
000→020→120
000→010→110→120
000→001→021→121→120
000→002→022→122→120

Step 2: Use the above paths between 000 and 120 to obtain the parallel paths between 0000 and 0120 following the construction of Case 2 of Theorem 1. This results in the following paths:

0000→0100→0120
0000→0200→0220→0120
0000→0020→0120
0000→0010→0110→0120
0000→0001→0021→0121→0120

0000→0002→0022→0122→0120
0000→1000→1100→1120→0120
0000→2000→2100→2120→0120

Finally, we present the parallel paths construction method (illustrated in the above example) in the form of an the following algorithm.

<u>function</u> PATHS ($a_{n-1}a_{n-2} \ldots a_0, b_{n-1} b_{n-2} \ldots b_0, n, k$)
(returns a set of $n(k-1)$ parallel paths between $a_{n-1}a_{n-2} \ldots a_0$ and $b_{n-1}b_{n-2} \ldots b_0$ in Q_k^n) .

<u>begin</u>
<u>if</u> n = 1 <u>then</u> return $\{a_0 \rightarrow \_b_0\} \cup \{ a_0 \rightarrow i \rightarrow b_0$, for $0 \le i < k$ and $i \ne a_0$ and $i \ne b_0\}$
<u>else</u> if $a_{n-1} = b_{n-1}$ <u>then</u>
<u>begin</u>
S_1 = PATHS($a_{n-2} \ldots a_0, b_{n-2} \ldots b_0, n-1, k$)
Let π_1 be a minimum length path in S_1
$S_2 = \varnothing$

<u>for</u> each path π in S_1 <u>do</u> $S_2 = S_2 \cup \{< a_{n-1}, \pi\_>\}$
<u>for</u> each i, $0 \le i < k$ and $i \ne a_{n-1}$ <u>do</u>
$S_2 = S_2 \cup \{a_{n-1} \ldots a_0 \rightarrow ia_{n-2} \ldots a_0 \parallel <i,\pi_1> \parallel ib_{n-2} \ldots b_0 \rightarrow b_{n-1} \ldots b_0\}$
return S_2
<u>end</u>
<u>else</u> <u>begin</u>
S_1 = PATHS($a_{n-2} \ldots a_0, b_{n-2} \ldots b_0, n-1, k$)

Let π_1 be a minimum length path in S_1
$S_2 = \{a_{n-1} \ldots a_0 \rightarrow b_{n-1}a_{n-2} \ldots a_0 \parallel <b_{n-1}, \pi_1>\}$
<u>for</u> each i, $0 \le i < k$ and $i \ne a_{n-1}$ and $i \ne b_{n-1}$ <u>do</u>
$S_2 = S_2 \cup \{a_{n-1} \ldots a_0 \rightarrow ia_{n-2} \ldots a_0 \parallel <i, \pi_1> \parallel ib_{n-2} \ldots b_0 \rightarrow b_{n-1} \ldots b_0\}$
$S_2 = S_2 \cup \{<a_{n-1}, \pi_1> \parallel a_{n-1}b_{n-2} \ldots b_0 \rightarrow b_{n-1} \ldots b_0\}$
<u>for</u> each path π in S_1 such that $\pi \ne \pi_1$ <u>do</u>
$S_2 = S_2 \cup \{<a_{n-1},\pi\text{-}1> \parallel a_{n-1} b'_{n-2} \ldots b'_0 \rightarrow b_{n-1} b'_{n-2} \ldots b'_0 \rightarrow b_{n-1} \ldots b_0\}$
(where $b'_{n-2} \ldots b'_0$ is the last intermediate node in π)
return S_2
<u>end</u>
<u>end</u>

Summary

In this paper, we have contributed to the study of the topological properties of the k-ary n-cube by presenting a simple algorithm (easy to implement) for constructing node-disjoint (parallel) paths between arbitrary nodes of the k-ary n-cube. In fact, we show how to construct efficiently a complete set of parallel paths (i.e. as many paths as the degree of the network) between any two nodes of a k-ary n-cube. Furthermore, each of the constructed paths is shown to be of length at most 2 plus the minimum distance between the two nodes.

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