BMOA Characterization with Families of Cauchy Transforms

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ABSTRACT: In this paper we prove a number of results on Cauchy transforms of generalized type given by Borel measures supported on the class of analytic functions mapping the unit disc into the unit disk.

Let \( \Delta = \{ z \in C : |z| < 1 \} \), \( \Gamma = \partial \Delta \) and let \( B ( \Gamma ) \) (equipped with the topology of uniform convergence on compact subsets) denote the set of functions \( \phi \) that are analytic in \( \Delta \) such that \( |\phi(z)| < 1 \) and \( \phi(0) = 0 \). Let \( M, N \) denote the sets of complex-valued Borel measures on \( \Gamma \) and \( B \) respectively. Here, \( M \) is equivalent to the subset of \( N \) consisting of all those measures supported on the set \( \{ x : |x| = 1 \} \).

For \( z \in \Delta \) and \( \alpha \geq 0 \), let \( A_\alpha \) denote the family of functions \( f \) for which there exists a measure \( \mu \in N \) such that

\[
f(z) = \begin{cases} \frac{1}{B(1-\phi(z))^\alpha} d\mu(\phi) & \text{for } \alpha > 0 \\ \int_B \log \frac{1}{1-\phi(z)} d\mu(\phi) + f(0) & \text{for } \alpha = 0 \end{cases}
\]  

The classes \( F_\alpha \) consisting of those functions \( g \) for which there exists a measure \( \mu \in M \) such that,

\[
g(z) = \begin{cases} \frac{1}{(1-xz)^\alpha} d\mu(x) & \text{for } \alpha > 0 \\ \int [ \log \frac{1}{1-xz} d\mu(x) + g(0) & \text{for } \alpha = 0 \end{cases}
\]

have been well studied (Hallenbeck et al, 1996; Hallenbeck and Samoilij, 1993; Hruscev and Vinogradov, 1981; Vinogradov, 1980). The classes \( F_\alpha \) are subsets of \( A_\alpha \) when the measures \( \mu \) in (1.1) are in \( M \).

The class \( A_\alpha \) is a Banach space with respect to the norm

\[
\| f \|_{A_\alpha} = \begin{cases} \inf \| \mu \| & \text{for } \alpha > 0 \\ \inf \| \mu \| + |f(0)| & \text{for } \alpha = 0 \end{cases}
\]
where $\mu$ varies over all measures in $\mathcal{N}$ for which the measures $\mu$ in (1.1) are in $\mathcal{M}$.

Clearly, for $f \in F_\alpha$, $\| f \|_{F_\alpha} \geq \| f \|_{A_\alpha}$. It is also known from (Brannan et al., 1973) that for $\alpha \geq 1$, $F_\alpha = A_\alpha$.

We will show in this paper that for $0 < \alpha < \beta$,

$$A_\alpha \subset A_\beta \quad \text{and} \quad \| f \|_{A_\alpha} \leq \| f \|_{A_\beta}$$

(1.4)

This generalizes similar results for $F_\alpha$ in (Hibschweiler and Nordgren, 1996).

We will also show that $A_\alpha = BMOA$ and that the norm $\| \cdot \|_{A_\alpha}$ is equivalent to well known $BMO$ norms. Furthermore we will show that, for all $n \geq 0$, $\| z^n \|_{A_\alpha} \leq k$ where the constant $k$ is independent of $n$ or $\alpha$.

The Classes $A_\alpha$

In this section we will establish for $0 \leq \alpha < \beta$ the relationship between $A_\alpha$ and $A_\beta$ as well as their respective norms.

**Theorem 1**: If $0 \leq \alpha < \beta$, then $A_\alpha \subset A_\beta$ and $\| f \|_{A_\alpha} \leq \| f \|_{A_\beta}$.

**Proof**: Note that since $A_\alpha = F_\alpha$ for $\alpha \geq 1$ (Brannan et al., 1973), and for $0 < \alpha < \beta$, $F_\alpha \subset F_\beta$ and $\| f \|_{F_\beta} \leq \| f \|_{F_\alpha}$ (Hibschweiler and Nordgren, 1996), then all we have to prove is the case $0 \leq \alpha < \beta < 1$.

(i) Let $f \in A_\alpha$ where $0 < \alpha < \beta$, then we can write

$$f(z) = \int_{\mathbb{R}} \frac{1}{(1 - \psi(z))^\alpha} d\mu(\psi).$$

(2.1)

Since $\frac{1}{(1 - z)^\alpha} \in F_\alpha \subset F_\beta$, we can write

$$\frac{1}{(1 - z)^\alpha} = \int_{\mathbb{R}} \frac{1}{(1 - \bar{x}z)^\beta} d\nu(x)$$

(2.2)

and

$$\left\| \frac{1}{(1 - z)^\alpha} \right\|_{F_\beta} \leq \left\| \frac{1}{(1 - z)^\beta} \right\|_{F_\alpha} = 1$$

(2.3)

Now by replacing $z$ in (2.2) by $\psi(z)$ and putting the result in (2.1) we get

$$f(z) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1 - \bar{x} \psi(z))^{\beta}} d\nu(x) d\mu(\psi).$$

(2.4)

Suppose, without loss of generality that $\nu$ is a positive measure and let

$$g_n(z) = \sum_{k=1}^{n} \frac{\nu_k}{(1 - \bar{x}_k z)^\beta}.$$
Then by (2.2)
\[ \int g_n(\psi)d\mu(\psi) \]
converges locally uniformly to
\[ f(z) = \int \int \frac{1}{(1 - \overline{\psi}(z))^\sigma} d\nu(x)d\mu(\psi). \]

Let \( \eta_n(\psi) = \sum_{k=1}^{n} \nu_k d\mu(\psi) \), then,
\[ \int g_n(\psi)d\mu(\psi) = \int \frac{1}{(1 - \psi(z))^\sigma} d\eta_n(\psi), \]
where \( ||\eta_n|| \leq ||\nu|| ||\mu|| \) for all \( n \). Hence by compactness, there exists a measure \( \sigma \), such that,
\[ f(z) = \int \frac{1}{(1 - \psi(z))^\sigma} d\sigma(\psi), \] (2.5)
which shows that \( f \in A_\sigma \). Furthermore, \( ||\sigma|| \leq ||\nu|| ||\mu|| \). Consequently,
\[ ||f||_{A_\nu} \leq ||\sigma|| \leq ||\mu|| ||\nu|| \] for all \( \mu \).

However since \( \mu \) and \( \nu \) are arbitrary measures that give (2.1) and (2.2) then,
\[ ||f||_{A_\nu} \leq \inf\{||\mu||\} \inf\{||\nu||\}. \]

Hence by (2.3)
\[ ||f||_{A_\nu} \leq ||f||_{A_\alpha}. \]

(ii) Now let \( f \in A_0 \). We want to show that \( f \in A_\alpha \) for any \( \alpha > 0 \). By definition,
\[ f(z) = \int \log \frac{1}{1 - \phi(z)} d\mu(\phi) + f(0) \] (2.6)

Since \( \log \frac{1}{1 - z} \in F_0 \subseteq F_\alpha \) (see Hibschweiler and MacGregor, 1989), then
\[ \log \frac{1}{1 - z} = \int \frac{1}{(1 - \overline{\phi}(z))^\alpha} d\nu(x) \]
where \( \nu \) depends only on \( \alpha \). Hence (2.6) becomes
\[ f(z) = \int \int \frac{1}{(1 - \overline{\phi}(z))^\alpha} d\nu(x)d\mu(\phi) + f(0) \] (2.7)
where the integral in (2.7) looks exactly like the one in (2.3) with $\alpha$ replacing $\beta$ and hence using an argument similar to the one in (i) we will get that

$$f(z) = \int_B \int I \frac{1}{(1 - \nu(z))^\alpha} d\sigma(w) + f(0)$$

(2.8)

which shows that $f \in A_\alpha$. Furthermore $\| \sigma \| \leq \| \nu \| + \| \mu \|$, hence

$$\| f \|_{A_\alpha} \leq \inf(\| \mu \|) + \| f(0) \| = \| f \|_{\alpha_0}$$

(2.9)

**Characterization of $A_\alpha$**

It is known (Garrett, 1980, p248) that a function $\phi \in \text{BMO}$ if and only if there exists functions $\phi_1$ and $\phi_2$ in $L^\infty$ such that

$$\phi = \phi_1 + \phi_2 + \alpha$$

where both $\| \phi_1 \|_\infty$ and $\| \phi_2 \|$ are less than $C \| \phi \|_\infty$, $C$ is a constant and $\| \cdot \|_\infty$ is the classical BMO norm (Garrett, 1980, p248).

Consequently $f \in \text{BMOA}$ if and only if three are analytic functions $f_1$ and $f_2$ such that

$$f = f_1 + f_2 + \alpha$$

(3.1)

where $\| \text{Re} f_1 \|_\infty \leq C$ and $\| \text{Im} f_2 \|_\infty \leq C$.

If we define on BMOA the norm

$$\| f \|^* = \inf\{ \| \text{Re} f_1 \|_\infty + \| \text{Im} f_2 \|_\infty : f = f_1 + f_2 + \alpha \}$$

(3.2)

then by (Garrett, 1980, p248), the norms $\| f \|^*$ and $\| f \|_{\alpha_0}$ are equivalent.

Now we have the following proposition which establishes a set equality between $A_\alpha$ and BMOA.

**Theorem 2: $A_\alpha = \text{BMOA}$**

**Proof:** Suppose that $f \in A_\alpha$ then according to (1.1) and (1.2) there exists a measure $\mu \in \mathcal{M}$ such that,

$$f(z) = \int_B \log \frac{1}{1 - \phi(z)} d\mu(\phi) + f(0)$$

(3.3)

Assume without loss of generality, that $\mu$ is a probability measure. Then $f$ is subordinate to $\log \frac{1}{1 - z} + f(0)$ and consequently by (3.1), $f \in \text{BMOA}$. The proof of the other inclusion follows from (3.1) and subordination.

**Theorem 3:** The norms $\| \cdot \|$ and $\| \cdot \|_{\alpha_0}$ are equivalent, namely there exists positive constant $c_1$ and $c_2$ such that

$$c_1 \| f \| \leq \| f \|_{\alpha_0} \leq c_2 \| f \|$$

(3.4)
**Proof:** Suppose \( f \in \text{BMOA} \), then \( f \) can be decomposed as in (3.1). Let \( d_1 \) denote \( \| \text{Re} f \|_{\alpha} \) and \( d_2 \) denote \( \| \text{Im} f \|_{\alpha} \). Then

\[
\frac{\pi}{2d_1} \left| \text{Im} f_1(z) \right| \leq \frac{\pi}{2}
\]

(3.5)

and

\[
\frac{\pi}{2d_2} \left| \text{Im} f_2(z) \right| \leq \frac{\pi}{2}
\]

(3.6)

for all \( z \in \Delta \). Consequently, by subordination

\[
if_1(z) = \frac{2d_1}{\pi} \left( \log \frac{1}{1 - \phi(z)} - \log \frac{1}{1 + \phi(z)} \right) + if_1(0)
\]

(3.7)

\[
f_2(z) = \frac{2d_2}{\pi} \left( \log \frac{1}{1 - \psi(z)} - \log \frac{1}{1 + \psi(z)} \right) + f_2(0)
\]

(3.8)

for all \( z \in \Delta \) and where \( \phi, \psi \in B \). Therefore

\[
\| f \|_{\alpha_0} \leq \frac{4}{\pi} (d_1 + d_2),
\]

(3.9)

and hence

\[
\| f \|_{\alpha_0} \leq \frac{4}{\pi} \| f \|_{\alpha} \leq c_2 \| f \|_{\alpha}.
\]

(3.10)

which gives the right inequality in (3.3).

Next, we show the left inequality. Let us write \( f \) as in (1.2) and assume without loss of generality that \( \mu \) is a positive measure. Then

\[
\left| \text{Im} f(z) \right| \leq c \| \mu \|.
\]

(3.11)

where \( c > 1 \). Thus

\[
\left| \text{Im} f(z) \right| \leq c \| f \|_{\alpha_0},
\]

(3.12)

and since

\[
\| \text{Im} f \|_{\alpha} \leq \| \text{Im} f \|,
\]

(3.13)

we have

\[
\| f \|_{\alpha} \leq k_1 \| \text{Im} f \|_{\alpha} \leq k_2 \| f \|_{\alpha_0} = \frac{1}{c_1} \| f \|_{\alpha_0},
\]

(3.14)

where \( c_1 = \frac{1}{k_2} \) and the left inequality in (3.13) follows by (Garrett, 1980, p235) and this concludes the proof.

**THEOREM 4:** \( \| z^n \|_{\alpha_0} \leq k \) for \( n \geq 1 \)

**Proof:** It is enough to show that \( \| z^n \|_{\alpha_0} \leq k \) for \( n \geq 1 \). Since we showed that \( \| \cdot \|_{\alpha} \) and \( \| \cdot \|_{\alpha_0} \) are equivalent, let us approximate \( \| z^n \|_{\alpha} \). It is known from ([2], p240) that
\[
\| f \|^2 \approx \sup_{\varphi} \int_{\mathbb{A}} \left| \nabla g \right|^2 \left( 1 - |z|^2 \right)^2 |\psi'(z)| \, dA
\]  
(3.15)

where \( \psi(z) = \frac{z + z_0}{1 + z_0 \bar{z}} \) is a Möbius transformation. Replace \( g \) in (3.15) by \( z^n \) and \( |z| \) by \( r \) to get,

\[
1 = \int_{\mathbb{A}} \left| \nabla (z^n) \right|^2 (1 - r^2) |\psi'(z)| \, dA \\
\leq \int_{\mathbb{A}} n^2 r^{2n-2} (1 - r^2) |\psi'(z)| \, dA \\
\leq \int_0^{2\pi} \int_0^1 n^2 r^{2n-2} (1 - r^2) |\psi'(z)| \, dr \, d\theta \\
\leq 2m^2 \int_0^1 r^{2n-2} (1 - r) \, dr, \quad \text{because } \int_0^{2\pi} |\psi'(z)| \, d\theta \leq 2\pi \\
\leq 2m^2 \left[ \frac{4\pi n^2}{(4n^2 - 1)} \right] \leq \frac{4\pi}{3} 
\]  
(3.16)

which gives us the desired result and completes the proof.

The following theorem is a direct consequence of Proposition 4.

**Theorem 5:** If \( f(z) = \sum_{n=0}^\infty a_n z^n \) is analytic and if \( \sum_{n=1}^\infty |a_n| < \infty \) then \( f \in A_0 \) for all \( \alpha \geq 0 \).

**Proof:** It is sufficient to prove that \( f \in A_0 \) since \( A_0 \subset A_\alpha \). To show that \( f \in A_0 \) all we have to show is that the norm \( \| f(z) \|_{A_0} \) is bounded.

\[
\| f(z) \|_{A_0} = \left\| \sum_{n=0}^\infty a_n z^n \right\|_{A_0} \leq \sum_{n=0}^\infty |a_n| \quad \| z^n \|_{A_0} \leq \frac{4\pi}{3} \sum_{n=0}^\infty |a_n| < \infty.
\]

**References**


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