*SQU Journal for Science*, 2017, 22(2), 114-119 DOI: http://dx.doi.org/10.24200/squjs.vol22iss2pp114-119
Sultan Qaboos University

# Finite Element Approximation of Variational Inequalities: An Algorithmic Approach

**Messaoud Boulbrachene**

*Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, PC 123, Al-Khoud, Muscat, Sultanate of Oman. Email: boulbrac@squ.edu.om.*

**ABSTRACT:** In this paper, we introduce a new method to analyze the convergence of the standard finite element method for elliptic variational inequalities with noncoercive operators (VI). The method consists of combining the so-called Bensoussan-Lions algorithm with the characterization of the solution, in both the continuous and discrete contexts, as fixed point of contraction. Optimal error estimates are then derived, first between the continuous algorithm and its finite element counterpart, and then between the true solution and the approximate solution.

**Keywords:** Variational inequalities, Algorithm, Finite element, $L^\infty$ - error estimate.

## تقريب العناصر المنتهية للتباينات المتفاوتة: المنهج الخوارزمي

**مسعود بولبراشن**

**الملخص:** نتطرق في هذا البحث إلى التحدث عن طريقة جديدة لتحليل نقطة إلتقاء العناصر المنتهية للتباينات البيضاوية المتفاوتة باستخدام المؤثرات غير الإلزامية. حيث تتكون هذه الطريقة من دمج خوارزمية بنسوزان ليونز مع تمييز الحلول في السياق المستمر والمنفصل كنقطة ثابتة من التقلص. كما تشتق تقديرات الأخطاء المثلى أولا من الخوارزمية المستمرة والعنصر النظير المنتهي، وانتهاءً بالحل الصحيح، والتقريبي.

**الكلمات المفتاحية:** التباينات المتفاوتة، الخوارزمية، العنصر المنتهي، تقدير الأخطاء.

## 1. Introduction

The theory of variational inequalities finds its roots in the work of Signorini [1] and Fichera [2] concerning unilateral problems. The mathematical foundation of the theory was widened by the invaluable contributions of Stampacchia [3] and then developed by the French and Italian schools (Stampacchia [4], Brezis [5], Mosco [6], Bensoussan-Lions [7]). It has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, finance, economics, and in social and pure and applied sciences.

This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of science and prove to be productive and innovative.

In this paper, we are concerned with the standard finite element approximation of the noncoercive problem associated with elliptic variational inequalities (VI): Find u $\in K$ such that

$$a(u,v-u) \geq (\mathrm{f}, \mathrm{v}-\mathrm{u}) \quad \forall v \in K \tag{1}$$

Here, $\Omega$ is a bounded domain of $\mathbb{R}^N$, with boundary $\Gamma$, f in $L^\infty(\Omega)$, and $\psi \in W^{2,\infty}(\Omega)$ such that $\partial\psi / \partial n \leq 0$ on $\Gamma$, (.,.) is the inner product in $L^2(\Omega)$, K is the closed convex set

$$K = \{v \in H^1(\Omega) \text{ such that } v \leq \psi\} \tag{2}$$

and $a(.,.)$ is the bilinear form defined by $\forall u,v \in H^1(\Omega)$,

$$a(u,v) = \int_{\Omega}\left( \sum_{j,k=1}^{n} a_{j,k} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} v + a_0(x)\,\mathrm{u}v \right) dx \ , \tag{3}$$

where the coefficients $a_{j,k}$ , $b_k$ $a_0$ ( $j,k = 1,...,\mathrm{N}$ ) are sufficiently smooth, and satisfy

$$\sum_{j,k=1}^{n} a_{j,k}\xi_j\xi_k \geq \alpha|\xi|^2 \ \ \forall \xi \in \mathbb{R}^N \ , \ \alpha > 0 \ , \ \forall x \in \Omega \tag{4}$$

$$\mathrm{a}_0(\mathrm{x}) \geq \beta > 0 \ , \ \forall x \in \Omega \quad . \tag{5}$$

Denoting by $\mathrm{V}_h$ the finite element space consisting of continuous piecewise linear functions, $r_h$ the usual interpolation operator, and where

$$K_h = \{v \in \mathrm{V}_h \ \text{such that} \ v \leq r_h\psi\} \ , \tag{6}$$

we define the discrete counterpart of (.) by find $u_h \in K_h$ such that

$$a(u_h, v - u_h) \geq \ (f, v - u_h) \ \forall v \in K_h \quad . \tag{7}$$

In stochastic control problems, the coefficients $\mathrm{b}_\mathrm{k}(\mathrm{x}), \mathrm{a}_0(\mathrm{x})$ can be such that the bilinear form (3) does not satisfy the usual coercivity assumption, making the problem under consideration noncoercive. Hence, in order to handle the noncoercive situation, we consider the equivalent VIs:

$$b(u, v - u) \geq (f + \lambda u, v - u) \ \forall v \in K \tag{8}$$

and

$$b(u_h, v - u_h) \geq \ (f + \lambda u_h, v - u_h) \ \forall v \in K_h \tag{9}$$

for the continuous and discrete problems, respectively, where $\lambda > 0$ is large enough so that the new bilinear form

$$b(u,v) = a(u,v) + \lambda(u,v) \tag{10}$$

is strongly coercive on $H^1(\Omega)$, that is,

$$b(v,v) \geq \delta \|v\|^2_{H^1(\Omega)}, \ \delta > 0 \quad . \tag{11}$$

The standard finite element approximation for VIs with noncoercive operators was first studied in [9], where an optimal error estimate was established by means of a subsolution method. In a recent work [10], we developed a new method to carry out the approximation of the same problem, combining the Bensoussan-Lions (BL) algorithm and the concept of subsolutions. In the present paper, we instead combine, in both the continuous and discrete contexts, the BL- Algorithm with the characterization of the solution as a fixed point of a contraction. The novelty that results from this combination resides in the fact that the generated algorithm is both monotone and geometrically convergent with a rate of convergence depending explicitly on the coercivity parameter $\lambda$ .

Combining the geometrical convergence results with standard finite element maximum norm error estimates for elliptic VIs, we first establish an error estimate between the continuous algorithm and its finite element version, and then between the exact solution and the finite element approximate.

An outline of the paper is as follows: in section 2, we recall some qualitative properties and standard finite element error estimate results for elliptic coercive VIs. In section 3, we establish, in both the continuous and discrete cases, the geometrical convergence of the algorithm. Finally, in section 4, we give the finite element error analysis.

## 2. Preliminaries

Let $g$ in $L^\infty(\Omega)$ and $\omega = \partial(\mathrm{g})$ be the solution of the following coercive VI: Find $\omega \in K$ such that

$$b(\omega, v - \omega) \geq (g, v - \omega) \ \forall v \in K \tag{12}$$

**Lemma 1.** [7] Let $g$ and $\tilde{g}$ in $L^\infty(\Omega)$, and ω and $\tilde{\omega}$ be the corresponding solutions to VI (12). Then $g \geq \tilde{g}$ implies $\omega \geq \tilde{\omega}$.

**Proposition 1.** Under conditions of Lemma 1, we have

$$\|\omega - \tilde{\omega}\|_\infty \leq \frac{1}{\lambda + \beta}\|g - \tilde{g}\|_\infty$$

**Proof.** For $g, \tilde{g}$ in $L^\infty(\Omega)$ we consider $\xi = \partial(g)$ and $\xi = \partial(\tilde{g})$ the corresponding solution to VI (.). Let also $\Phi = \dfrac{1}{\lambda+\beta}\|g - \tilde{g}\|_\infty$. Then, since

$$\begin{aligned} g &\le \tilde{g} + \|g - \tilde{g}\|_\infty, \\ &\le \tilde{g} + ((a_0(x)+\lambda)/(\lambda+\beta) \\ &\le \tilde{g} + (a_0(x)+\lambda)\Phi \quad (because \quad a_0(x) \ge \beta > 0) \end{aligned}$$

Making use of lemma 1, we get

$$\partial(g,\psi) = \partial(g + (a_0(x)+\lambda)\Phi, \psi)$$

On the other hand, one has

$$\partial(\tilde{g},\psi) + \Phi = \xi + \Phi = \partial(g + (a_0(x)+\lambda)\Phi; \psi + \Phi)$$

Indeed,

$$\begin{aligned} b(\xi+\Phi,\ v+\Phi-(\xi+F)) &\ge b(\xi,\ v-\xi) + ((a_0(x)+\lambda)\Phi,\ v-\xi) \\ &\ge (\ \tilde{g},\ v-\xi) + ((a_0(x)+\lambda),\ v-\xi) \end{aligned}$$

so

$$b(\xi+\Phi,\ v+\Phi-(\xi+\Phi)) \ge (\tilde{g} + (a_0(x)+\lambda)\Phi,\ v+\Phi-(\xi+\Phi))$$

$$\xi+\Phi \le \psi+\Phi;\ v+\Phi \le v+\Phi$$

Therefore,

$$\partial(g + (a_0(x)+\lambda)\Phi;\ \psi) \le \partial(g + (a_0(x)+\lambda)\Phi;\ \psi+\Phi)$$

That is

$$\xi \le \xi + \Phi$$

The roles of $g$ and $\tilde{g}$ being symmetric, we similarly get

$$\xi \le \xi + \Phi$$

and the result follows.

Now, let $\omega_h = \partial_h(g) \in K_h$ denote the solution of the discrete counterpart of VI (12), that is,

$$b(\omega_h,\ v-\omega_h) \ge (g,\ v-\omega_h)\ \forall v \in K_h$$

**Remark 1.** Lemma 1 and proposition 1 stay true in the discrete case, provided the stiffness matrix is an M-Matrix (this will be thoroughly explained in section 3).

**Theorem 1.** [11] **T**here exists a constant $C$ independent of $h$ such that

$$\|\omega - \omega_h\|_\infty \le Ch^2 |\log h|^2 \tag{13}$$

## 3. Algorithms

### 3.1 The continuous algorithm

Consider the following mapping $T : L^\infty(\Omega) \to K$ such that $w \to Tw = \xi$, where $\xi$ is the unique solution of the following coercive VI

$$b(\xi,\ v-\xi) = (f + \lambda w,\ v-\xi)\ \forall v \in K$$

Now, starting from $u^0 = \psi$, we define the sequence $(u^n)_{n\ge1}$ by

$$u^n = Tu^{n-1},\ \forall n \ge 1 \tag{14}$$

such that each iterate $u^n$ solves the coercive VI

$$b(u^n,\ v-u^n) = (f + \lambda u^{n-1},\ v-u^n)\ \forall v \in K \tag{15}$$

Note that, thanks to lemma 1, the above sequence is monotonic decreasing.

**Theorem 2.** Under conditions of proposition1, the mapping T is a contraction. Therefore, its unique fixed point coincides with the solution of VI (1), and we have the error bound

$$\|u^n - u\|_\infty \le \frac{\rho^n}{1-\rho}\|u^0 - u^1\|_\infty$$

**Proof.** Let $w$ and $\tilde{w}$ in $L^{\infty}(\Omega)$, $\xi = \partial(f + \lambda w), \xi = \partial(f + \lambda\tilde{w})$ be the corresponding solutions to VI (.). Then, making use of proposition 1, we have

$$\begin{aligned}\|Tw - T\tilde{w}\|_{\infty} &= \|\xi - \xi\|_{\infty} \\ &\leq \frac{1}{\lambda+\beta}\left\| f + \lambda w - (f + \lambda\tilde{w})\right\|_{\infty} \\ &\leq \frac{\lambda}{\lambda+\beta}\|w - \tilde{w}\|_{\infty}\end{aligned}$$

which yields the contraction of $T$. The error bound follows straightforward from the fact that T is a contraction.

### 3.2 The discrete Algorithm

For the sake of simplicity, we suppose that $\Omega$ is polyhedral. We then consider a regular and quasi-uniform triangulation $\tau_h$ of $\overline{\Omega}$, consisting of $n$ simplices $\kappa$. Denote by $h$, the mesh size of $\tau_h$, with $h_\kappa$ being the diameter of $\kappa$. For each $\kappa \in \tau_h$, denote by $P_1(\kappa)$, the set of polynomials on $\kappa$ with degree no more than 1. The $P_1$ conforming finite element space is given by

$$V_h = \{v \in H^1(\Omega) \cap C(\overline{\Omega}) : v/\kappa \in P_1(\kappa)\} \quad . \tag{16}$$

Let $M_i$, $1 \leq i \leq m(h)$ denote the vertices of the triangulation $\tau_h$, and let $\varphi_i$, $1 \leq i \leq m(h)$ denote the functions of $V_h$ which satisfy

$$\varphi_i(M_j) = \delta_{ij},\ 1 \leq i, j \leq m(h)$$

so that the functions $\varphi_i$ form a basis of $V_h$. For every $v \in H^1(\Omega) \cap C(\overline{\Omega})$, the function

$$r_h v(x) = \sum_{i=1}^{m(h)} v(M_i)\varphi_i(x) \tag{17}$$

represents the interpolate of $v$ over $\tau_h$.

The convergence analysis of the discrete algorithm will require the monotonicity of the stiffness matrix.

**Definition 1.** A real $d \times d$ matrix $C = (c_{ij})$ with $c_{ij} \leq 0$, $\forall i \neq j$, $1 \leq i, j \leq d$, is called an M-Matrix if $C$ is nonsingular and $C^{-1} \geq 0$ (i.e., all entries of its inverse are nonnegative).

Denote by $B$ the matrix with generic coefficient

$$b_{ij} = a(\varphi_i, \varphi_j) + \lambda\int_{\Omega}\varphi_i\varphi_j dx, 1 \leq i, j \leq m(h) \quad . \tag{18}$$

Since the bilinear form $b(.,.)$ is coercive, then the matrix $B$ is positive definite and $b_{ii} > 0$, $1 \leq i \leq m(h)$. Furthermore, if the matrix $(a_{jk})$ involved in the bilinear form (3) is symmetric, then mesh conditions for which the off-diagonal entries of $B$ satisfy $b_{ij} \leq 0, \forall i \neq j$.

Lemma 2 [13],[14] The matrix $B$ is an M- Matrix.

**Theorem 3.** [9] Under conditions of lemma 2, the discrete VI (7) has a unique solution.

Let us now consider the mapping $T_h : L^{\infty}(\Omega) \to K_h$ such that $w \to T_h w = \xi_h$, where $\xi_h$ is the unique solution of the following discrete coercive VI:

$$b(\xi_h, v - \xi_h) \geq (f + \lambda w, v - \xi_h) \forall v \in K_h .$$

Starting from $u_h^0 = r_h\psi$, we define the discrete algorithm by

$$u_h^n = T_h u_h^{n-1},\ \forall n \geq 1 \tag{19}$$

such that each iterate $u_h^n$ solves the coercive VI

$$b(u_h^n, v - u_h^n) \geq (f + \lambda u_h^{n-1}, v - u_h^n) \forall v \in K_h \quad . \tag{20}$$

As, in the continuous case, thanks to Remark 1, the above sequence is monotonic decreasing.

**Theorem 4.** Under conditions of lemma 2, the mapping $T_h$ is a contraction. Therefore, its unique fixed point coincides with the solution of VI (7) and we have

$$\left\|u_h^n - u_h\right\|_\infty \leq \frac{\rho^n}{1-\rho}\left\|u_h^0 - u_h^1\right\|_\infty . \tag{21}$$

**Proof.** Similar to that of Theorem 3.

## 4. $L^\infty(\Omega)$ - Error Analysis

This section is devoted to proving the main results of this paper. Next, we shall estimate the error in the maximum norm between the nth iterates $u^n$ of the algorithm and its finite element counterpart $u_h^n$. For that, let us first introduce the following sequence of coercive VIs. Indeed, let us define the sequence $\left(\bar{u}_h^n\right)_{n\geq 1}$ such that $\bar{u}_h^n$ solves the discrete VI

$$b(\bar{u}_h^n,\ v - u_h^n) = (f + \lambda u^{n-1},\ v - \bar{u}_h^n) \forall v \in K_h \tag{22}$$

where $u^n$ is the nth iterate of the continuous algorithm. From now on, $C$ will denote a constant independent of both $n$ and $h$.

**Lemma3.** We have

$$\left\|u^n - \bar{u}_h^n\right\|_\infty \leq Ch^2 |\log h|^2 \tag{23}$$

**Theorem 5.** We have

$$\left\|u^n - u_h^n\right\|_\infty \leq Ch^2 |\log h|^2 \tag{24}$$

**Proof.** We proceed by induction. Indeed, let $\theta(h) = Ch^2 |\log h|^2$. Then, using standard error estimate in the maximum norm, we have

$$\left\|u^0 - u_h^0\right\|_\infty = \left\|\psi - r_h\psi\right\|_\infty \leq Ch^2 |\log h|^2$$

which, combined with Theorem 4 and estimate (23), yields

$$\begin{aligned}
\left\|u^1 - u_h^1\right\|_\infty &\leq \left\|u^1 - \bar{u}_h^1\right\|_\infty + \left\|\bar{u}_h^1 - u_h^1\right\|_\infty \\
&\leq \theta(h) + \left\|T_h u^0 - T_h u_h^0\right\|_\infty \\
&\leq \theta(h) + \rho\left\|u^0 - u_h^0\right\|_\infty \\
&\leq (1+\rho)\theta(h) = \frac{1-\rho^2}{1-\rho}\theta(h)
\end{aligned}$$

Now assume that

$$\left\|u^{n-1} - u_h^{n-1}\right\|_\infty \leq \frac{1-\rho^n}{1-\rho}\theta(h)$$

Then, combining again Theorem 4 and estimate (23), we get

$$\begin{aligned}
\left\|u^n - u_h^n\right\|_\infty &\leq \left\|u^n - \bar{u}_h^n\right\|_\infty + \left\|\bar{u}_h^n - u_h^n\right\|_\infty \\
&\leq \theta(h) + \left\|T_h u^{n-1} - T_h u_h^{n-1}\right\|_\infty \\
&\leq \theta(h) + \rho\left\|u^{n-1} - u_h^{n-1}\right\|_\infty \\
&\leq \theta(h) + \frac{1-\rho^n}{1-\rho}\theta(h) \\
&\leq \left(1 + \rho\frac{1-\rho^n}{1-\rho}\right)\theta(h)
\end{aligned}$$

**Theorem 6.** We have

$$\left\| u - u_h \right\|_\infty \leq Ch^2 \,|\log h\,|^2 \quad . \tag{24}$$

**Proof.** Indeed, combining Theorems 2, 4, and 5 we have

$$\left\| u - u_h \right\|_\infty \leq \left\| u - u^n \right\|_\infty + \left\| u^n - u_h^n \right\|_\infty + \left\| u_h^n - u_h \right\|_\infty$$

$$\leq \frac{\rho^n}{1-\rho} \left\| u^0 - u^1 \right\|_\infty + Ch^2 \,|\log h\,|^2 + \frac{\rho^n}{1-\rho} \left\| u_h^0 - u_h^1 \right\|_\infty$$

So passing to the limit, as $n \to \infty$, we get

$$\left\| u - u_h \right\|_\infty \leq Ch^2 \,|\log h\,|^2 .$$

## 5. Conclusion

Based on the constructive Bensoussan-Lions Algorithm and the Banach fixed point principle, we have derived error estimate in the maximum norm of the standard finite element approximation of elliptic variational inequalities with non coercive operators. This new approach turns out to be successful and may be extended, in a future work, to system of variational inequalities related to HJB equations.

## References


1. Signorini, A. On some static issues of continuous systems. *Annals of the Scuola Normale Superiore di Pisa,* 1933, **2**, 231-251.
2. Fichera, G. Elastostatic problems with unilateral constraints; the Signorini problem with ambiguous contingency conditions. *National Academy of the Lincei*, 1964,**7**, 91-140
3. Stampacchia, G. Bilinear coercive forms on convex sets. *Comptes Rendus de l' academie des Sciences, Paris*, 1964, **258**, 4413-4416 .
4. Lions, J.L. and Stampacchia, G. Variational inequalities. *Communication in Pure and Applied Mathematics.,* 1967, **20**, 493-519.
5. Brezis, H. Equations and nonlinear in dual vector spaces. *Annals of Fourier Institute, Grenoble*, 1968, **18**, 115-175.
6. Mosco, U. Implicit variational problems and quasi-variational inequalities. *Lecture notes in Mathematics*, *Springer, Berlin* 1975, **543**, 83-156.
7. Bensoussan, A. and Lions, J.L., Application of variational inequalities to stochastic control problems. *Gauthiers-Villars*, 1982, Paris.
8. Rodrigues, J.F. Obstacle problems in mathematical physics. 1987, North-Holland, New York.
9. Cortey-Dumont, P. On variational inequalities with non coercive operators, *RAIRO*, 1985, **2**, 195-212.
10. Boulbrachene, M. On The finite element approximation of variational inequalities with noncoercive operators. *Numerical Functional Analysis and Optimization,* 2015, **36**, 1107-1121.
11. Cortey Dumont, P. On the finite element approximation in the $L^\infty$ norm of variational inequalities with nonlinear operators. *Numerische Mathematik*, 1985,**47**, 45-57.
12. Atkinson, W. and Han, K. Theoretical Numerical Analysis, ***3rd*** *Edition*, 2010, *Springer*.
13. Lu, C., Huang. and Qiu, J. Maximum principle in linear finite element approximations of anisotropic diffusion--convection--reaction problems. *Numerische Mathematik*, 2014, **3**, 515-537.
14. Vejchodsky, T. The discrete maximum principle for Galerkin solutions of elliptic problems. *European Journal of Mathematics*, 2012, **10**, 5-43.


Received 30 March 2017
Accepted 25 September 2017