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On the Computation of the Maximum Uncertainty Volume of Stable Polynomials

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حساب قيمة أكبر حجم لتغير المعاملات بواسطة طرق الحلول المثلى

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خلاصة : نقترح طريقة تعتمد على الحلول المثلى من أجل إيجاد أكبر قدر من تغير المعاملات في الأنظمة . ندرس كلا من الأنظمة الخطية مع وجود صندوق من الأخطاء في المعاملات وكذلك معين من الدوال. بعد تقديم الطريقة المقترحة نستعرض استخدامها عن طريق تطبيقها على بعض الأمثلة.

ABSTRACT: In this paper we propose a non-linear optimization based approach for the computation of the stability region for uncertain polynomials. Both box of polynomials and diamond of polynomials are addressed. Examples are presented as an illustration.

KEYWORDS: Optimization, Uncertain Polynomials, Box Polynomials, Diamond Polynomials.

1. Introduction

The computation of the largest perturbations of a given system without violating stability is of great interest for analysis and controller design. Considerable work has been reported in the literature focusing on this issue. The work in Barmish (1984) makes use of the Kharitonov result (Kharitonov, 1978) to find the smallest destabilizing perturbations in the parameters by examining the four Hurwitz testing matrices. The work assumes equal perturbations to all the parameters. In Chapellat *et al*. (1988) a procedure for obtaining the largest stability ball is proposed. The authors in Blanchini *et al.* (1998) proposed to find the minimum destabilizing volume for uncertain systems, and in Kim *et al*. (1988) the case of bivariate polynomials was studied. In Hollot (1988), it was proposed to use the markov parameters of a polynomial to study the stability margin, and Djaferis (1989) studied polynomials with linear uncertainty. The diamond of polynomials case was studied in Barmish *et al*. (1992).

In this study we make use of the link established by Mansour *et al*. (1992) between the celebrated Kharitonov results and the positive definiteness of the associated Hermite matrices (Anderson, 1972), to produce a non-linear programming approach for obtaining the largest volume in the parameter space around a stable nominal polynomial.

The statement of the problem is given in the following section, followed by the proposed scheme in section three. Examples are presented in section four, and section five concludes the paper.

2. Statement of the problem

Definition 1. N(s) is called an interval polynomial if it is given by

$$N(s) = l_0 + l_1 s + ... + l_{n-1} s^{n-1} + l_n s^n$$

with the parameters in the given intervals bounded below by the minimum value , and above by the maximum value (i.e. $l_i \in [l_{\bar{i}}; \vec{l}_i])$

Definition 2. p(s,d) represents diamond of polynomials if

$$P(s,d) = d_0 + d_1 s + ... + d_{n-1} s^{n-1} + d_n s^n \tag{1}$$

with real coefficients d_i known to lie in the *(n*+1)-dimensional diamond *centered* at $d^* = (d^*_0 .... d^*_n)$. This means that $d = [d_0\ d_1\ d_2 ...... d_n]$ is a possible value for the parameters if $d \in D_n$ where

$$D_r = \{d : | d_0 - d_0^* | + | d_1 - d_1^* | + ... | d_n - d_n^* | \leq r\} \tag{2}$$

and r is the radius of the diamond.

Definition 3. The Hermite matrix H associated with a given polynomial of order n, $p(s,d)$ is an n by n symmetric matrix with entries given by

$$\begin{aligned} H_{ij} &= \sum_{k=1}^{i} (-1)^{k+i} d_{n-k+1} d_{n-i-j+k} && ; j \geq i, and\ i + j\ even \\ &= H_{ji} && ; j < i, and\ i + j\ even \\ &= 0 && ; i + j\ is\ odd \end{aligned} \tag{3}$$

We now present the stability results for both: the box of polynomials, and a diamond of polynomials.

Lemma 1: The interval polynomial N(s) is stable (Kharitonov, 1978) for all l_i if the following four polynomials, of order n, are stable:

$$N_1(s) = l_{-0} + l_{-1} s + \bar{l}_2 s^2 + \bar{l}_3 s^3 + ...$$
$$N_2(s) = l_{-0} + l_{-1} s + \bar{l}_2 s^2 + l_{-3} s^3 + ...$$
$$N_3(s) = \bar{l}_0 + l_{-1} s + l_{-2} s^2 + \bar{l}_3 s^3 + ...$$
$$N_4(s) = \bar{l}_0 + \bar{l}_1 s + l_{-2} s^2 + l_{-3} s^3 + ...$$

Lemma 2: (Barmish *et al*., 1992) The *diamond* of polynomials (1) is stable for all d, $d \in D_r$ if the following eight polynomials are stable:

$$\begin{aligned} P^1(s,d) &= P(s,d^*) + r \\ P^2(s,d) &= P(s,d^*) - r \\ P^3(s,d) &= P(s,d^*) + rs \\ P^4(s,d) &= P(s,d^*) - rs \\ P^5(s,d) &= P(s,d^*) + rs^{n-1} \\ P^6(s,d) &= P(s,d^*) - rs^{-n-1} \\ P^7(s,d) &= P(s,d^*) + rs^n \\ P^8(s,d) &= P(s,d^*) - rs^n \end{aligned}$$

Lemma 3: (Anderson, 1972) A polynomial is stable if the associated Hermite matrix is positive definite.

It is the objective of this paper to find the maximum range of the parameters which maintain the stability of the polynomials. Both the interval and the diamond like uncertainty will be addressed. In our development, Lemmas 1 and 3 will be used to formulate the problem for finding the maximum volume of uncertainty, for interval polynomials, as a optimization problem. Also, lemmas 2 and 3 will be used for the formulation of the case of a diamond of polynomials.

3. An optimization-based approach

3.1 Interval polynomials

In order to find the maximum range, we propose to find the interval ranges which maximize the volume of the box of uncertainty, V_b. To fix notation, let a^*_i be the nominal value of parameter a_i. Let the parameter a_i belong to the range $[a^*{}_i - \delta_i^l, a^*{}_i - \delta_i^u]$. The width of this range is equal to $\delta_i^l + \delta_i^u$. The volume of the box of uncertainty is given by

$$V_b = \prod_{i=0}^{n}(\delta_i^l + \delta_i^u)$$

Our proposed scheme is to maximize V_{b1} subject to the stability constraints given by the positive definiteness of the four Hermite matrices corresponding to the Kharitonov polynomials. The problem of finding the maximum stabilizing volume may then be presented as

$$\underset{\delta_i^l, \delta_i^u}{Max} V_b = \prod_{i=0}^{5}(\delta_i^l + \delta_i^u)$$

subject to the constrains given by

$$H_i > 0 \qquad i = 1, 2, 3, 4$$

3.2 Diamond polynomials

A similar approach may be suggested for the study of stability radius of a diamond of polynomials. We need to maximize the volume of the diamond, V_d, subject to the positive definiteness of the eight Hermite matrices associated with the eight polynomials given in lemma 2. The volume of the diamond, V_d, with radius r, is a fraction multiple of the volume of the box V_b, with length $2r$. So the NLP problem to solve is

$$\underset{r}{Max} V_b = (2r)^{n+1}$$

subject to the constraints represented by

$$H_i > 0, \qquad i = 1,2........8$$

with H_i the Hermite matrix associated with the i^{th} polynomial given in lemma 2. Note that H_i are functions of one parameter, r, only.

The problem of maximizing the volume V_b is a non-linear optimization problem. A global maximum is not guaranteed. however, the proposed approach gives the flexibility of finding the effect of individual parameters and it also can put some constraints on the uncertainties being symmetric or not etc.

The optimization can be performed using the Matlab optimization routine called “constr”. This routine allows the user to provide the objective function, and the constraints of the optimization problem and produces the maximum value after the optimization is done.

4. Examples

4.1 Example 1

Consider the following fourth order stable polynomial (Barmish 1984)

$$P(s) = s^4 + 5s^3 + 8s^2 + 8s + 3$$

We would like to find the values of $\delta_i^{\,l}$ and $\delta_i^{\,u}$ for the interval polynomial which maximizes

$$V_b = (\delta_0^{\,l} + \delta_0^{\,u})(\delta_1^{\,l} + \delta_1^{\,u})(\delta_2^{\,l} + \delta_2^{\,u})(\delta_3^{\,l} + \delta_3^{\,u})$$

subject to $H_i > 0$, $i = 1, ..4$, which are the Hermite matrices associated with the four Kharitonov polynomials.

If we restrict the changes $\delta_i^{\,l}$ and $\delta_i^{\,u}$ to be the same, then we have the situation usually studied in the literature. This is also the case studied in (Barmish 1984). The result of the optimization scheme is $\delta_i^{\,l} = \delta_i^{\,u} = 1.81$, $i = 0, 1, 2, 3$ (This represents a one parameter optimization problem). The result is in agreement with the result reported by Barmish.

Now we relax the constraints and assume that $\delta_i^{\,l} = \delta_i^{\,u}$, but they are different for different values of i. The resulting stability volume is given by the following parameter intervals:

[[1,1], [2.318, 7.682], [6.33, 9.67], [5.93, 10.07], [1.71.4.29]]

and with the removal of the above restriction, we have the following intervals:

[[1,1], [5, 20.28], [8, 103365.4], [8, 30.027], [0,3]]

4.2 Example 2

Consider now the example studied in (Blanchini *el at*. 1998). The example they studied is a sixth order polynomial given by

$$p(s) = s^6 + 4s^5 + 4s^4 + 6s^3 + 3s^2 + 2s + 0.5$$

The one parameter optimization case resulted in the values of $\delta_i^{\,l} = \delta_i^{\,u} = 0.0512$, and with different $\delta_i^{\,l}$ and $\delta_i^{\,u}$ for different i, the resulting intervals are given by:

[[1,1], [3.91, 4.09], [3.91, 4.09], 5.88, 6.12], [2.99, 3.01], [1.96, 2.04], [.47, .53]]

4.3 Example 3

Here we consider a diamond of polynomials studied in (Barmish *et al.*, 1992) and given by:

$$p(s) = s^4 + 3s^3 + 6.49s^2 + 7.98s + 3.45$$

The objective is to find the maximum value of r, such that the polynomial (1) is stable for all parameters in D_n given in (2). Our proposed method resulted in the diamond D_r given by

$$D_n = \{d : |d_a\text{-}3.49| + |d_1 - 7.98| + |d_2 - 6.49| + |d_3 - 3| + |d_4 - 1| \le 0.9593\}|$$

This is very close, but larger compared to the value of $r = 0.9467$ obtained in Barmish *et al.* (1992).

5. Conclusions

An optimization-based approach is proposed for obtaining the largest volume of perturbations of the plant parameters, the use of which is illustrated via examples. The case of polynomials with parameters which are affine in some uncertain parameters can be addressed using the proposed approach.

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