Characterizations of K- Semimetric Spaces

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ABSTRACT: In this paper, we prove, for a space $X$, the following are equivalent:
1. $X$ is a $\omega \Delta$ space with a regular-$G_\delta$-diagonal,
2. $X$ is a $\omega \Delta_2$ space with a regular-$G_\delta$-diagonal,
3. $X$ is a semi-developable space with $G_\delta(3)$-diagonal,
4. $X$ is a $\omega \Delta_1$-space with a $G_\delta(3)$-diagonal,
5. $X$ is a $\omega \Delta_2$-space with a $G_\delta(3)$-diagonal,
6. $X$ is a $\omega \Delta$-space with a $G_\delta(3)$-diagonal,
7. $X$ is a semi-developable space with $G_\delta^*(2)$-diagonal,
8. $X$ is a semimetrizable, $c$-stratifiable space,
9. $X$ is a $c$-Nagata $\beta$-space,
10. $X$ is a $K$-semimetrizable.

KEYWORDS: $\omega \Delta$-space, semi-developable space, $K$-semimetrizable space, $\beta$-space, $G_\delta^*(2)$-diagonal, $G_\delta(3)$-diagonal, regular-$G_\delta$-diagonal, semi-stratifiable, $c$-semi-stratifiable.

1. Introduction

A space $X$ is semimetrizable if there exists a real valued function $d$ on $X \times X$ such that
1. $d(x,y) = d(y,x) \geq 0$.
2. $d(x,y) = 0$ if and only if $x = y$.
3. for $M \subset X$, $x \in \overline{M}$ if and only if $d(x,M) = \inf \{d(x,y) : y \in M\} = 0$. If in addition, $d$ satisfies.
4. $d(H,K) > 0$ whenever $H$ and $K$ are disjoint compact subsets of $X$, then $X$ is said to be $K$-semimetrizable (Arhangel'skii, 1966).

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of covers of a space $X$.

1. Suppose $\{G_n\}_{n \in \mathbb{N}}$ satisfies the following property: if, $x_n \in \text{st}(x,G_n)$, then the sequence $\langle x_n \rangle$ has a cluster point.
(a) If, for each \( n \in \mathbb{N} \), \( G_n \) is an open cover of \( X \), then \( X \) is called a \( \omega \Delta \)-space (Borges, 1968).
(b) If, for each \( n \in \mathbb{N} \), \( st(x,G_n) \) is an open subset of \( X \), then \( X \) is called a \( \omega \Delta_1 \)-space (Gittings, 1975).
(c) If, for each \( n \in \mathbb{N} \), \( x \in \text{Int} \ st (x,G_n) \), then \( X \) is called a \( \omega \Delta_2 \)-space (Gittings, 1975).

2. If for each \( x \in X \), \( \{ st (x,G_n) \}_{n \in \mathbb{N}} \) is a local base at \( x \), then \( X \) is called a semi-developable space. If in addition, for each \( n \in \mathbb{N} \), \( st(x,G_n) \) is an open subset of \( X \), then \( X \) is called a semi-developable space.

3. If, for each \( n \in \mathbb{N} \), \( G_n \) is an open cover of \( X \) and for each \( x \in X \), \( \bigcap_n st^3 (x,G_n) = \{ x \} \), then \( X \) has a \( G_\delta \) (3)-diagonal.

4. If, for each \( n \in \mathbb{N} \), \( G_n \) is an open cover of \( X \) and for each \( x \in X \), \( \bigcap_n st^2 (x,G_n) = \{ x \} \), then \( X \) has a \( G_\delta^* \) (2)-diagonal.

5. If, for each \( n \in \mathbb{N} \), \( st(x,G_n) \) is an open subset of \( X \) and for each \( x \in X \), \( \bigcap_n st (x,G_n) = \{ x \} \), then \( X \) has a \( S_2 \) -diagonal.

6. If, for each \( n \in \mathbb{N} \), \( x \in \text{Int} \ st (x,G_n) \) and for each \( x \in X \), \( \bigcap_n st (x,G_n) = \{ x \} \), then \( X \) has a \( \alpha_2 \)-diagonal.

7. If, for each \( n \in \mathbb{N} \), \( G_n \) is an open cover of \( X \) and for any pair of distinct points \( x,y \in X \), there exist neighborhoods \( U \) and \( V \) of \( x \) and \( y \), respectively, and \( n \in \mathbb{N} \), such that \( st(U,G_n) \cap V = \phi \), equivalently, \( st(V,G_n) \cap U = \phi \), then \( X \) has a regular- \( G_\delta \)-diagonal.

A COC-map (= countable open covering map) for a topological space \( X \) is a function from \( \mathbb{N} \times X \) into the topology of \( X \) such that for every \( x \in X \), and \( n \in \mathbb{N} \), \( x \in g(n,x) \) and \( g(n,x) \subseteq g(n,x) \). A space \( X \) is called \( \beta \)-space if \( X \) has a COC-map \( g \) such that if \( x \in g(n,x) \) for every \( n \in \mathbb{N} \), then the sequence \( \{ x_n \} \) has a cluster point.

A space \( X \) is called \( q \)-space if \( X \) has a COC-map \( g \) such that if \( x_n \in g(n,x) \) for every \( n \in \mathbb{N} \), then the sequence \( \{ x_n \} \) has a cluster point.

A space \( X \) is called \( c \)-semi-stratifiable (Martin, 1973) \((c \)-stratifiable) if there is a sequence \( \{ g(n,x) \} \) of open neighborhoods of \( x \) such that for each compact set \( K \subset X \), if \( g(n,K) = \bigcup \{ g(n,x) : x \in K \} \), then \( \bigcap \{ g(n,K) : n \geq 1 \} = K \left( \bigcap \{ g(n,K) : n \geq 1 \} = K \right) \).

The \( COC \)-map \( g : \mathbb{N} \times X \rightarrow \tau \) is called a \( c \)-semi-stratification \((c \)-stratification) of \( X \). A space \( X \) is called \( c \)-Nagata if it is first countable, \( c \)-stratifiable space.

Throughout this paper, all spaces are assumed to be \( T_2 \)-spaces unless otherwise stated explicitly. The letter \( \mathbb{N} \) always denotes the set of all positive integers.

2. Main results

\textbf{Lemma 1 :} Every space with a \( G_\delta \) (3)-diagonal has a \( G_\delta^* \) (2)-diagonal.

\textit{Proof.} Let \( \{ G_n \} \) be a \( G_\delta \) (3)-diagonal sequence for \( X \). We want to prove that \( \bigcap_{n \in \mathbb{N}} st^2 (x,G_n) = \{ x \} \) for every \( x \in X \). Suppose we have \( q \in \bigcap_{n \in \mathbb{N}} st^2 (x,G_n) \). For every open set \( U \) such that \( q \in U \) and for each \( n \in \mathbb{N} \)

\[ st^2 (x,G_n) \cap U \neq \phi. \]
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In particular, if $G \in G_n$ is such that $q \in G$ then $st^2(x,G_n) \cap G \neq \emptyset$. So, $q \in st^3(x,G_n)$. As this holds for all $n$, it follows that $x = q$.

**Lemma 2:** Any space with a $G_\delta^*$ (2)-diagonal is a c-stratifiable space.

**Proof.** Let $\{G_n\}$ be a sequence of open covers of a space $X$ such that $\bigcap_{n \in \mathbb{N}} st^2(x,G_n) = \{x\}$. Define a COC-map $g$ by

$$g(n,x) = st(x,G_n).$$

We must prove that $\bigcap g(n,K) = K$ for any compact subset of $X$.

Let $p \notin K$. Then, for each $k \in K$, there exists an integer $n(k)$ such that $p \notin st^2(k,G_{n(k)})$. Therefore there is an open set $U(k)$ containing $p$ such that $U(k) \cap st^2(k,G_{n(k)}) = \emptyset$. Since $K$ is compact, we can find a finite number of points $k_1, k_2, \ldots, k_r$ of $K$ such that $\{st(k_j,G_{n(k)}): 1 \leq j \leq r\}$ covers $K$. Let $n = \max\{n(k_i): 1 \leq i \leq r\}$, and $U = \bigcap_{k \in K} U(k)$. Then

$$U \cap st(k,G_n) = \emptyset.$$

That is, $U \cap g(n,K) = \emptyset$. This implies $p \notin g(n,K)$.

**Theorem 1:** Every $\omega \Delta_1$-space with $S_2$-diagonal is an $\omega$-semidevelopable space.

**Proof.** Let $\{G_n\}_{n \in \mathbb{N}}$ be a countable family of covers of a space $X$ illustrating that $X$ is a $\omega \Delta_1$-space. Since $X$ has an $S_2$-diagonal, there exists a sequence $\{\nu_n : n \in \mathbb{N}\}$ of covers of $X$ such that, for each $x \in X$ and $n \in \mathbb{N}$, $st(x,\nu_n)$ is an open subset of $X$ and $\bigcap_{n \in \mathbb{N}} st(x,\nu_n) = \{x\}$. For each $n \in \mathbb{N}$, let

$$u_n = \left\{U : U = \left(\bigcap_{i=1}^n G_i\right) \cap \left(\bigcap_{i=1}^n V_i\right), G_i, V_i \in \nu_i, i = 1, 2, \ldots, n\right\}.$$

It is easy to see that $u_{n+1}$ refines $u_n$ for all $n \in \mathbb{N}$ and that, for each $x \in X$, $\bigcap_{n \in \mathbb{N}} st(x,u_n) = \{x\}$. Furthermore, for each $x \in X$ and $n \in \mathbb{N}$

$$st(x,u_n) = \left(\bigcap_{i=1}^n st(x,G_i)\right) \cap \left(\bigcap_{i=1}^n st(x,\nu_i)\right)$$

and thus $st(x,u_n)$ is an open subset of $X$. Also, it is easy to check that $\langle u_n : n \in \mathbb{N}\rangle$ is a $\omega \Delta_1$-sequence for $X$.

It remains to show that $\langle u_n : n \in \mathbb{N}\rangle$ is a semi-development for $X$. Suppose instead that $\langle u_n : n \in \mathbb{N}\rangle$ is not a semi-development for $X$. Then there is a point $x$, an open neighborhood $W$ of $x$, and a sequence $\langle x_n \rangle$ such that for all $n$, $x_n \in st(x,u_n)$ and $x_n \notin W$. Since $\langle u_n : n \in \mathbb{N}\rangle$ is a $\omega \Delta_1$-sequence for $X$, the sequence $\langle x_n \rangle$ has a cluster point $p$. Clearly $p \notin W$ so $p \neq x$. By choice of $\langle \nu_n : n \in \mathbb{N}\rangle$, there are $k$ in $\mathbb{N}$ and a neighborhood $V$ of $p$ such that $V \cap st(x,\nu_k) = \emptyset$. Now for $n \geq k$, $x_n \in st(x,u_n) \subset st(x,u_k) \subset st(x,\nu_k)$ so $x_n \notin V$. This contradicts the fact that $p$ is a cluster point of $\langle x_n \rangle$. Thus $\langle u_n : n \in \mathbb{N}\rangle$ is a semi-development for $X$.

**Theorem 2:** The following are equivalent for a regular $\omega \Delta_1$-space $X$:

1. $X$ is semimetrizable;
2. $X$ is semi-stratifiable;
3. $X$ is $\theta$-refinable and has a $G_\delta$-diagonal;
4. $X$ has a $G^*\delta$-diagonal;
(5) $X$ has $\alpha_2$-diagonal.

(6) $X$ is semidevelopmentable.

Proof. The only implications requiring comment are (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1). To prove (5) $\Rightarrow$ (6), let $\{G_n\}$ be a countable family of covers of $X$ illustrating that $X$ is a $\omega\Delta_2$-space. Let 

$$\langle v_n : n \in \mathbb{N} \rangle$$

be an $\alpha_2$-sequence for $X$. Let the sequence $\langle u_n : n \in \mathbb{N} \rangle$ be defined as in the proof of Theorem 2.3. Since for each $x \in X$ and $n \in \mathbb{N}$, 

$$\text{Int}(x, u_n) = \left( \bigcap_{n=1}^{\infty} \text{Int}(x, G_n) \right) \cap \left( \bigcap_{n=1}^{\infty} \text{Int}(x, v_n) \right),$$

we have $x \in \text{Int}(x, u_n)$. It follows, exactly as before, that $\langle u_n : n \in \mathbb{N} \rangle$ is a semi-development for $X$. The implication (6) $\Rightarrow$ (1) follows from (Alexander, 1971), Theorem 1.3.

Theorem 3:

For a space $X$, the following are equivalent:

1. $X$ is a $\omega\Delta_1$-space with a regular-$G_\delta$-diagonal,
2. $X$ is a $\omega\Delta_2$-space with a regular-$G_\delta$-diagonal,
3. $X$ is a semi-developable space with $G_\delta (3)$-diagonal,
4. $X$ is a $\omega\Delta_1$-space with a $G_\delta (3)$-diagonal,
5. $X$ is a $\omega\Delta_2$-space with a $G_\delta (3)$-diagonal,
6. $X$ is a $\omega\Delta_2$-space with a $G_\delta (3)$-diagonal,
7. $X$ is a semi-developable space with $G_\delta^* (3)$-diagonal,
8. $X$ is a semi-stratifiable, $c$-stratifiable space,
9. $X$ is a $c$-Nagata $\beta$-space,
10. $X$ is a $K$-semimetrizable.

Proof. It is clear that $1 \Rightarrow 2$, $3 \Rightarrow 4$, $4 \Rightarrow 5$, $8 \Rightarrow 9$.

The implication $5 \Rightarrow 6$ follows by Lemma 2.5 and since every $\omega\Delta_2$-space is a $q, \beta$-space. The implication $6 \Rightarrow 7$ follows by facts every $\beta$-space with a $G_\delta^* (3)$-diagonal is a semi-stratifiable space, every $q$-space with a $G_\delta^* (3)$-diagonal is first countable and every first countable, semi-stratifiable space is a semimetrizable.

The implication $7 \Rightarrow 8$ follows by Lemma 2.2 and since every $T_\theta$-semi-developable space is a semimetrizable.

The implication $9 \Rightarrow 8$ follows by facts every $c$-stratifiable, $\beta$-space is semi-stratifiable and every first countable, semi-stratifiable space is a semimetrizable.

$1 \Rightarrow 8$ follows by Lemma 2.2, Theorem 2.3.

For $2 \Rightarrow 3$. Suppose that $X$ is a $\omega\Delta_2$-space with a regular-$G_\delta$-diagonal. Every space with a regular-$G_\delta$-diagonal has a $G_\delta^* (3)$-diagonal. By Theorem 2.4, $X$ is a semi-developable space. Let $\{G_n\}$ be a semi-development and regular-$G_\delta$-diagonal-sequence. To see that $G_n$ satisfies the $G_\delta (3)$-diagonal-sequence, let $x \neq y$ points in $X$, $U$ and $V$ open sets containing $x$ and $y$ respectively, and $n_0$ an integer such that if $n > n_0$, then no member of $G_n$ meets both $U$ and $V$. Let $n_1$ and $n_2$ be integers such that $st(x, G_{n_1}) \subset U$ and $st(y, G_{n_2}) \subset V$. $N = \max\{n_0, n_1, n_2\}$. Then no member of $G_n$ meets both $st(x, G_n)$ and $st(y, G_n)$. Thus $y \not\in st^3(y, G_n)$.

For $10 \iff 3$. Let $G_n = \{1/n \text{ sphere centered at } x\}$. It is clear that $\{G_n\}$ is a sequence of covers of $X$ and $y \in st(x, G_n)$ if and only if $d(x, y) < 1/n$. Therefore $\{G_n\}$ is a semidevelopment for $X$. Now let $G_n = \{\text{interior of } 1/n \text{ sphere centered at } x\}$. It is clear that $\{G_n\}$ is a sequence of open covers of $X$ and if $y \in st(x, G_n)$ then $d(x, y) < 1/n$. If there exist distinct points $x$ and
such that \( y \in st^3(x,G_n) \) for all \( n \in \mathbb{N} \), then there are sequences \( \{x_n\} \) and \( \{y_n\} \) such that

\[ x_n \in st(x,G_n), y_n \in st(y,G_n) \quad \text{and} \quad y_n \in st(x_n,G_n). \]

Let \( K_1 = \{x\} \cup \{x_n : n \in \omega\} \) and \( K_2 = \{y\} \cup \{y_n : n \in \omega\} \). We may assume \( K_1 \cap K_2 = \phi \) with both sets compact. But \( d(K_1, K_2) = 0 \), a contradiction.

Conversely, let \( G_n \) be a semi-development and \( G_\delta(3) \)-diagonal-sequence for \( X \). Define a semimetric \( d \) on \( X \) by \( d(x,y) = 1/\inf \{j \in \mathbb{N} : x \notin st(y,G_j)\} \). From the definition \( x \in st(y,G_n) \) if and only if \( d(x,y) < 1/n \). Assume there exist disjoint compacta \( K \) and \( H \) such that \( d(K,H) = 0 \). We can find two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( K \) and \( H \) respectively, such that \( d(x_n,y_n) < 1/n \). Note that \( X \) is sequential and \( T_2 \) so that \( \{x_n\} \) and \( \{y_n\} \) have convergent subsequences. Let \( \{x_{n_i}\} \) and \( \{y_{n_i}\} \) be subsequences of \( \{x_n\} \) and \( \{y_n\} \) converging to \( x \) and \( y \), respectively. Without loss of generality, we may assume \( d(x,x_{n_i}) < 1/i \) and \( d(y,y_{n_i}) < 1/i \) for each \( i \in \mathbb{N} \). Since \( d(x_{n_i},y_{n_i}) < 1/i \), it follows that there is no \( k \) such that \( y \notin st^3(x,G_k) \). This contradiction completes the proof.

**References**


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