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Solution of a Nonlinear Functional Equation

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حل معادلة دالية غير خطية

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خلاصة : في هذا البحث نقدم تعميم لنظرية ريستيسكي (2004) و الخاصة بحل معادلة دالية غير خطية. يعتمد البرهان علي تعريف البارامتر λ في المجموعة الاختيارية Λ مع طريقة المصفوفات لحل مثل هذا النوع من المعادلات.

ABSTRACT: In the present paper a generalization of a theorem of I.B. Risteski (2004) concerning the solution of a nonlinear functional equation is given. The proof is based on a parametric approach by introducing a parameter λ in an arbitrary set Λ, and on a matrix method for solving linear functional equations.

KEYWORDS: Nonlinear functional equation; Symmetric group.

1. Introduction

Functional equations find applications in biology, social sciences, engineering, as well as in many other branches of mathematics and a great number of such applications can be found in the monograph by Aczel (1966). This has led to considerable interest in the study of functional equations and has given rise to numerous articles and monographs on this subject.

The present paper is devoted to the study of a nonlinear functional equation which generalizes the quadratic complex vector functional equation solved by I.B. Risteski (2004). To the best of our knowledge, up to now this type of nonlinear functional equation has not been considered in the literature. We carried out our research to shed some light on this field of nonlinear functional equations. The results presented here and in (Risteski, 2004) supplement and extend some of the results in (Risteski *et al* 1999, 2000a, 2000b).

In order to prepare the background for our study, we present the result of I.B. Risteski (2004). Let V be a complex finite-dimensional vector space and let there exist mappings $f, g: V^n \to V$. $\mathbf{Z}_i$ $(1 \le i \le 2n)$

will denote vectors in the vector space V. Multiplication of two arbitrary vectors $\mathbf{U}=\left(u_1,u_2,\ldots,u_n\right)^T$ and $\mathbf{V}=\left(v_1,v_2,\ldots,v_n\right)^T$ in V is defined as $\mathbf{UV}=\left(u_1v_1,u_2v_2,\ldots,u_nv_n\right)^T$. I.B. Risteski (2004) gave the following result.

Theorem 1. *The general solution of the nonlinear complex vector functional equation*

$$\begin{aligned} &f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_n\right)f\left(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n}\right)\\ &=f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1}\right)f\left(\mathbf{Z}_n,\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n}\right)\\ &+f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2}\right)f\left(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\ldots,\mathbf{Z}_{2n}\right)\\ &+\cdots+f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n}\right)f\left(\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-1},\mathbf{Z}_n\right) \qquad \left(n\geq 2\right) \end{aligned} \tag{1}$$

is given by

$$f\left(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n\right)=\begin{vmatrix} F_1\left(\mathbf{U}_1\right) & F_1\left(\mathbf{U}_2\right) & \cdots & F_1\left(\mathbf{U}_n\right)\\ F_2\left(\mathbf{U}_1\right) & F_2\left(\mathbf{U}_2\right) & \cdots & F_2\left(\mathbf{U}_n\right)\\ \vdots & & & \\ F_n\left(\mathbf{U}_1\right) & F_n\left(\mathbf{U}_2\right) & \cdots & F_n\left(\mathbf{U}_n\right) \end{vmatrix}, \tag{2}$$

where F_i $\left(1\leq i\leq n\right)$ *are arbitrary functions in* V.

In the next section we shall consider a slightly generalized version of (1). In the proof of the theorem we shall use techniques developed in Risteski (2002) and Risteski and Covachev (2000, 2001). On the other hand, the method used in Risteski (2004) cannot be used in the proof of this theorem without imposing a very restrictive assumption on the equation considered.

2. Statement of the Problem

In this part of our paper, we shall consider the generalized nonlinear complex vector functional equation

$$\begin{aligned} &f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_n\right)g\left(\mathbf{Z}_{n+1},\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n}\right)\\ &=f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+1}\right)g\left(\mathbf{Z}_n,\mathbf{Z}_{n+2},\ldots,\mathbf{Z}_{2n}\right)\\ &+f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{n+2}\right)g\left(\mathbf{Z}_{n+1},\mathbf{Z}_n,\mathbf{Z}_{n+3},\ldots,\mathbf{Z}_{2n}\right)\\ &+\cdots+f\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_{2n}\right)g\left(\mathbf{Z}_{n+1},\ldots,\mathbf{Z}_{2n-1},\mathbf{Z}_n\right) \qquad \left(n\geq 2\right). \end{aligned} \tag{3}$$

If $f\equiv g$, equation (3) is reduced to (1).

It is easy to see that if a component of f is identically 0, then the corresponding component of g may be arbitrary. Similarly, if a component of g is identically 0, then the corresponding component of f may be arbitrary. So we need to consider only solutions $\left(f,g\right)$ of equation (3) for which no component of f or g is identically 0. Thus we may suppose that (3) is a scalar functional equation. Moreover, the arguments $\mathbf{Z}_i$ $\left(1\leq i\leq 2n\right)$ may belong to an arbitrary set V (with at least $n+1$ distinct elements). Furthermore, it is easily seen that the function f depends on the arguments $\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}\right)\in V^{n-1}$ as on a parameter. So

we can slightly generalize equation (3) by introducing a parameter λ in an arbitrary set Λ instead of $\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}\right)\in V^{n-1}$. For convenience we write $\mathbf{Z}_i$ instead of $\mathbf{Z}_{n+i}$ $\left(0\le i\le n\right)$.
Thus we consider the equation

$$f\left(\lambda,\mathbf{Z}_0\right)g\left(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)=f\left(\lambda,\mathbf{Z}_1\right)g\left(\mathbf{Z}_0,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)$$

$$+f\left(\lambda,\mathbf{Z}_2\right)g\left(\mathbf{Z}_1,\mathbf{Z}_0,\mathbf{Z}_3,\ldots,\mathbf{Z}_n\right)+\cdots+f\left(\lambda,\mathbf{Z}_n\right)g\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_0\right)\quad\left(n\ge 2\right), \tag{4}$$

where $f:\ \Lambda\times V\to\mathbb{C}$ and $g:\ V^n\to\mathbb{C}$.

3. Main Result

Theorem 2. *Each solution* $\left(f,g\right)$ *of the nonlinear functional equation* (4), *such that none of the functions* f *and* g *is identically* 0, *is given by*

$$f\left(\lambda,\mathbf{U}\right)=\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{U}\right), \tag{5}$$

$$g\left(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n\right)=\begin{vmatrix} G_1\left(\mathbf{U}_1\right) & G_1\left(\mathbf{U}_2\right) & \cdots & G_1\left(\mathbf{U}_n\right)\\ G_2\left(\mathbf{U}_1\right) & G_2\left(\mathbf{U}_2\right) & \cdots & G_2\left(\mathbf{U}_n\right)\\ \vdots & & & \\ G_n\left(\mathbf{U}_1\right) & G_n\left(\mathbf{U}_2\right) & \cdots & G_n\left(\mathbf{U}_n\right)\end{vmatrix}, \tag{6}$$

where $F_i\colon \Lambda\to\mathbb{C}$ *and* $G_i\colon V\to\mathbb{C}$ $\left(1\le i\le n\right)$ *are arbitrary functions in* V.

Proof: Let us introduce the function

$$F\left(\lambda,\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_n\right)=f\left(\lambda,\mathbf{Z}_0\right)g\left(\mathbf{Z}_1,\ldots,\mathbf{Z}_n\right). \tag{7}$$

It satisfies the linear functional equation

$$F\left(\lambda,\mathbf{Z}_0,\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)=F\left(\lambda,\mathbf{Z}_1,\mathbf{Z}_0,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right) \tag{8}$$

$$+F\left(\lambda,\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_0,\mathbf{Z}_3,\ldots,\mathbf{Z}_n\right)+\cdots+F\left(\lambda,\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1},\mathbf{Z}_0\right).$$

We will not solve equation (8). Instead, we will just find the form of the solutions of (8) which can be represented as (7). To this end, we use a method similar to that in Risteski (2002) and in Risteski and Covachev (2001).

Below we will denote by S_n the *symmetric group* of degree n (or symmetric group on n letters, see Feyzioğlu (1990)). This is the group of all one-to-one mappings of the set $\left\{1,2,\ldots,n\right\}$ onto itself, and its elements are called *permutations* (of $1,2,\ldots,n$). Here, for our convenience, we denote by S_{n+1} the group of all one-to-one mappings of the set $\left\{0,1,2,\ldots,n\right\}$ onto itself.

Let $\alpha\in S_{n+1}$. This means that $\left(\alpha\left(0\right),\alpha\left(1\right),\ldots,\alpha\left(n\right)\right)$ is a permutation of the indices $\left(0,1,\ldots,n\right)$. Then equation (8) is equivalent to the linear homogeneous system

$$F\left(\lambda,\mathbf{Z}_{\alpha(0)},\mathbf{Z}_{\alpha(1)},\mathbf{Z}_{\alpha(2)},\ldots,\mathbf{Z}_{\alpha(n)}\right)-F\left(\lambda,\mathbf{Z}_{\alpha(1)},\mathbf{Z}_{\alpha(0)},\mathbf{Z}_{\alpha(2)},\ldots,\mathbf{Z}_{\alpha(n)}\right) \tag{9}$$

$$-F\left(\lambda, \mathbf{Z}_{\alpha(2)}, \mathbf{Z}_{\alpha(1)}, \mathbf{Z}_{\alpha(0)}, \mathbf{Z}_{\alpha(3)}, \ldots, \mathbf{Z}_{\alpha(n)}\right) - \cdots$$
$$-F\left(\lambda, \mathbf{Z}_{\alpha(n)}, \mathbf{Z}_{\alpha(1)}, \ldots, \mathbf{Z}_{\alpha(n-1)}, \mathbf{Z}_{\alpha(0)}\right) = 0, \qquad \alpha \in S_{n+1}.$$

This system of $(n+1)!$ equations has a nontrivial solution if and only if its determinant is 0. We will not show that this determinant is 0, but we will use the form of the nontrivial solution and eventually obtain a solution of (8) admitting the factorization (7).

According to Risteski (2002) and Risteski and Covachev (2001), a possible nontrivial solution of (8) has the form

$$F(\lambda, \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{\alpha \in S_{n+1}} C_\alpha H\left(\lambda, \mathbf{Z}_{\alpha(0)}, \mathbf{Z}_{\alpha(1)}, \ldots, \mathbf{Z}_{\alpha(n)}\right), \tag{10}$$

where $H: \Lambda \times V^{n+1} \to \mathbb{C}$ is an arbitrary function, and C_α are complex constants (which may also depend on λ).

However, it is easy to see that if $F(\lambda, \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ has the form (7), then $H(\lambda, \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ must admit the factorization

$$H(\lambda, \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_n) = C(\lambda) G_0(\mathbf{Z}_0) G_1(\mathbf{Z}_1) \cdots G_n(\mathbf{Z}_n), \tag{11}$$

where $C: \Lambda \to \mathbb{C}$ and $G_i: V \to \mathbb{C}$ $(0 \le i \le n)$ are arbitrary functions.

Thus the representation (10) takes the form

$$F(\lambda, \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{\alpha \in S_{n+1}} C_\alpha(\lambda) G_0\left(\mathbf{Z}_{\alpha(0)}\right) G_1\left(\mathbf{Z}_{\alpha(1)}\right) \cdots G_n\left(\mathbf{Z}_{\alpha(n)}\right), \tag{12}$$

where we have denoted $C_\alpha(\lambda) = C_\alpha C(\lambda)$.

By virtue of (7) we must have

$$f(\lambda, \mathbf{Z}_0) g(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{\alpha \in S_{n+1}} C_\alpha(\lambda) G_0\left(\mathbf{Z}_{\alpha(0)}\right) G_1\left(\mathbf{Z}_{\alpha(1)}\right) \cdots G_n\left(\mathbf{Z}_{\alpha(n)}\right). \tag{13}$$

Since g is not identically 0, there exists an n-tuple $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in V^n$ such that $g(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \ne 0$. We set $\mathbf{Z}_i = \mathbf{A}_i$ $(i = 1, \ldots, n)$ in (13) and find

$$f(\lambda, \mathbf{Z}_0) g(\mathbf{A}_1, \ldots, \mathbf{A}_n) = \sum_{\alpha \in S_{n+1}} C_\alpha(\lambda) G_{\alpha^{-1}(0)}(\mathbf{Z}_0) \prod_{i=1}^{n} G_{\alpha^{-1}(i)}(\mathbf{A}_i), \tag{14}$$

i.e., $f(\lambda, \mathbf{Z}_0)$ can be represented as

$$f(\lambda, \mathbf{Z}_0) = \sum_{i=0}^{n} D_i(\lambda) G_i(\mathbf{Z}_0). \tag{15}$$

Now equation (13) can be written in the form

$$\sum_{i=0}^{n} D_i(\lambda) G_i(\mathbf{Z}_0) g(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{i=0}^{n} G_i(\mathbf{Z}_0) \left(\sum_{\alpha \in S_{n+1}} C_\alpha(\lambda) \prod_{j=1}^{n} G_{\alpha^{-1}(j)}(\mathbf{Z}_j) \right). \tag{16}$$

From (16) it follows that

$$g\left(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)=\sum_{\alpha\in S_{n+1}}\frac{C_\alpha\left(\lambda\right)}{D_i\left(\lambda\right)}\prod_{j=1}^{n}G_{\alpha^{-1}(j)}\left(\mathbf{Z}_j\right),\quad i=0,1,\ldots,n. \tag{17}$$

Now the quotients $C_\alpha\left(\lambda\right)/D_i\left(\lambda\right)$ must all be independent of λ. Moreover, if we compare two of these representations of $g\left(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)$, for $i=0$ and for some $i\neq 0$, and we give suitable values to all but one of the variables $\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n$, we find that $G_0\left(\mathbf{Z}\right)$ is expressed as a linear combination of $G_1\left(\mathbf{Z}\right),\ldots,G_n\left(\mathbf{Z}\right)$.

We can then write

$$f\left(\lambda,\mathbf{Z}_0\right)=\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{Z}_0\right), \tag{18}$$

$$g\left(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n\right)=\sum_{\alpha\in S_n}c_\alpha\prod_{j=1}^{n}G_j\left(\mathbf{Z}_{\alpha(j)}\right). \tag{19}$$

Equation (18) shows that, in fact, equation (5) is valid. It remains to prove that (up to a constant factor) equation (6) also holds. To this end we will determine the constants c_α (up to a common factor) so that the pair $\left(f,g\right)$ given by (18) and (19) satisfies equation (4):

$$\begin{aligned}&\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{Z}_0\right)\sum_{\alpha\in S_n}c_\alpha\prod_{j=1}^{n}G_j\left(\mathbf{Z}_{\alpha(j)}\right)\\&=\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{Z}_1\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(1)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(1)}G_j\left(\mathbf{Z}_{\alpha(j)}\right)\\&+\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{Z}_2\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(2)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(2)}G_j\left(\mathbf{Z}_{\alpha(j)}\right)+\cdots\\&+\sum_{i=1}^{n}F_i\left(\lambda\right)G_i\left(\mathbf{Z}_n\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(n)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(n)}G_j\left(\mathbf{Z}_{\alpha(j)}\right).\end{aligned}$$

Thus for each $i=1,2,\ldots,n$ we must have

$$\begin{aligned}G_i\left(\mathbf{Z}_0\right)\sum_{\alpha\in S_n}c_\alpha\prod_{j=1}^{n}G_j\left(\mathbf{Z}_{\alpha(j)}\right)&=G_i\left(\mathbf{Z}_1\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(1)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(1)}G_j\left(\mathbf{Z}_{\alpha(j)}\right)\\&+G_i\left(\mathbf{Z}_2\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(2)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(2)}G_j\left(\mathbf{Z}_{\alpha(j)}\right)+\cdots\\&+G_i\left(\mathbf{Z}_n\right)\sum_{\alpha\in S_n}c_\alpha G_{\alpha^{-1}(n)}\left(\mathbf{Z}_0\right)\prod_{j\neq\alpha^{-1}(n)}G_j\left(\mathbf{Z}_{\alpha(j)}\right).\end{aligned} \tag{20}$$

It is easily seen that each term on the left cancels with some term on the right-hand side of (20). Indeed, let us take the term

$$G_i\left(\mathbf{Z}_0\right)c_\alpha G_i\left(\mathbf{Z}_{\alpha(i)}\right)\prod_{j\neq i}G_j\left(\mathbf{Z}_{\alpha(j)}\right)$$

for some fixed $\alpha \in S_n$. Suppose that $\alpha(i) = k$. Then this term is identical with the term on the right

$$G_i(\mathbf{Z}_k) c_\alpha G_{\alpha^{-1}(k)}(\mathbf{Z}_0) \prod_{j \neq \alpha^{-1}(k)} G_j(\mathbf{Z}_{\alpha(j)}).$$

Now we have to determine (up to a common factor) the coefficients c_α so that the remaining terms on the right-hand side of (20) cancel pairwise. The terms on the right

$$G_i(\mathbf{Z}_k) c_\alpha G_{\alpha^{-1}(k)}(\mathbf{Z}_0) \prod_{j \neq \alpha^{-1}(k)} G_j(\mathbf{Z}_{\alpha(j)})$$

and

$$G_i(\mathbf{Z}_\ell) c_\beta G_{\beta^{-1}(\ell)}(\mathbf{Z}_0) \prod_{j \neq \beta^{-1}(\ell)} G_j(\mathbf{Z}_{\beta(j)})$$

for $k \neq \ell$ cancel each other if and only if

$$\alpha^{-1}(k) = \beta^{-1}(\ell), \quad \beta(i) = k, \quad \alpha(i) = \ell,$$
$$\alpha(j) = \beta(j) \quad \forall j \neq \alpha^{-1}(k), \quad c_\alpha + c_\beta = 0.$$

This means that

$$\beta\alpha^{-1}(k) = \ell, \quad \beta\alpha^{-1}(\ell) = k, \quad \beta\alpha^{-1}(m) = m \quad \forall m \in \{1,2,\dots,n\} \setminus \{k,\ell\}.$$

Thus the permutation $\beta\alpha^{-1}$ is a transposition (exchanges the indices k and ℓ and keeps all other indices fixed), *i.e.*, β and α are permutations of different signature (one even and one odd). Consequently, we must have

$$c_\alpha = \begin{cases} c & \text{if } \alpha \text{ is even,} \\ -c & \text{if } \alpha \text{ is odd,} \end{cases}$$

for some constant $c \neq 0$. Thus

$$g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) = c \begin{vmatrix} G_1(\mathbf{Z}_1) & G_1(\mathbf{Z}_2) & \cdots & G_1(\mathbf{Z}_n) \\ G_2(\mathbf{Z}_1) & G_2(\mathbf{Z}_2) & \cdots & G_2(\mathbf{Z}_n) \\ \vdots & & & \\ G_n(\mathbf{Z}_1) & G_n(\mathbf{Z}_2) & \cdots & G_n(\mathbf{Z}_n) \end{vmatrix}.$$

Finally, we can replace the functions $F_i(\lambda)$ by $cF_i(\lambda)$ for $i = 1,2,\dots,n$.

Thus we have shown that any solution (f, g) of equation (4), such that none of the functions f and g is identically 0, must have the form (5) and (6). It remains to show that every pair of functions (f, g) of the form (5) and (6) is a solution of equation (4). Indeed, by virtue of the equations (5) and (6) we obtain

$$f(\lambda, \mathbf{Z}_0) g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) - f(\lambda, \mathbf{Z}_1) g(\mathbf{Z}_0, \mathbf{Z}_2, \dots, \mathbf{Z}_n)$$
$$- f(\lambda, \mathbf{Z}_2) g(\mathbf{Z}_1, \mathbf{Z}_0, \mathbf{Z}_3, \dots, \mathbf{Z}_n) - \cdots - f(\lambda, \mathbf{Z}_n) g(\mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}, \mathbf{Z}_0)$$

$$= \sum_{i=1}^{n} F_i(\lambda) \begin{vmatrix} G_i(\mathbf{Z}_0) & G_i(\mathbf{Z}_1) & \cdots & G_i(\mathbf{Z}_n) \\ G_1(\mathbf{Z}_0) & G_1(\mathbf{Z}_1) & \cdots & G_1(\mathbf{Z}_n) \\ \vdots & & & \\ G_i(\mathbf{Z}_0) & G_i(\mathbf{Z}_1) & \cdots & G_i(\mathbf{Z}_n) \\ \vdots & & & \\ G_n(\mathbf{Z}_0) & G_n(\mathbf{Z}_1) & \cdots & G_n(\mathbf{Z}_n) \end{vmatrix} = 0.$$

This completes the proof of Theorem 2.

Remark 3. It is clear that the solutions of equations (1) and (4) as given, respectively, by (2), (5) and (6), coincide if we assume that $\Lambda = V^{n-1}$ and $f \equiv g$.

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