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A New Approach to the Stabilization of the Wave Equation with Boundary Damping Control

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إقرار معادلة موجة بنظام تحكم حدي مخمد

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خلاصة : يهتم هذا البحث بدراسة مسألة إقرار نظام مكون من معادلة موجة في مجال محدود مع شروط حدية من نوع نومان. حتى تتم عملية الاقرار نقترح نظام تحكم مخمد ونبرهن أن هذا النوع من التحكم بإمكانه أن يجعل حلول معادلة الموجة تتقارب نحو موضع إستقرار. وبالإضافة إلى ذلك ، نبرهن أن موضع الاستقرار يتعلق بالشروط الأساسية فقط.

ABSTRACT: This paper deals with boundary feedback stabilization of a system, which consists of a wave equation in a bounded domain of $\mathbb{R}^n$, with Neumann boundary conditions. To stabilize the system, we propose a boundary feedback law involving only a damping term. Then using a new energy function, we show that the solutions of the system asymptotically converge to a stationary position, which depends on the initial data. Similar results were announced without proof in (Chentouf and Boudellioua, 2004).

KEYWORDS: Wave equation, Neumann condition, boundary damping control, energy function, asymptotic behavior.

1. Introduction

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. Given a partition (Γ_0, Γ_1) of Γ, consider the following wave equation:

$$y_{tt}(x,t) - \Delta y(x,t) = 0, \quad \text{in} \quad \Omega \times (0,\infty) \tag{1.1}$$

with Neumann boundary conditions and initial conditions

$$\begin{cases} \dfrac{\partial y}{\partial \upsilon}(x,t)=0, & on \quad \Gamma_0 \times (0,\infty), \\ \dfrac{\partial y}{\partial \upsilon}(x,t)=U(t), & on \quad \Gamma_1 \times (0,\infty), \\ y(x,0)=y_0(x),\ y_t(x,0)=z_0(x), & \end{cases} \tag{1.2}$$

where υ is the unit normal of Γ pointing towards the exterior of Ω and U is a feedback law to be determined. Note that Γ_1 is supposed to be nonempty whereas Γ_0 may be empty.

As in Chentouf and Boudellioua (2004), we propose a feedback depending only on a damping term, that is,

$$U(t)=-a(x)y_t(x,t), \qquad (x,t)\in \Gamma_1 \times (0,\infty) \tag{1.3}$$

where $a \in L^{\infty}(\Gamma_1)$ satisfies $a(x) \geq a_0 > 0$ for any $x \in \Gamma_1$. Then, it is proved that the solutions of the closed loop system asymptotically tend towards a constant depending on the initial data y_0 and z_0.

There is a rich literature concerning the stabilization problem of the wave equation (see for instance Bardos *et al*. (1992), Chen (1979a, 1979b, 1981a, 1981b), Komornik (1994), Lax *et al*. (1963), Lagnese (1988), Lions (1988a, 1988b), Morawetz (1975), Quinn and Russell (1977), Triggiani (1975) and the references therein). In all references cited above, at least one of the following conditions is assumed to be satisfied:

- The equation (1.1) involves also the displacement term *y*.
- The stabilizing feedback law $U(t)$ contains not only a boundary dissipation term y_t but also the boundary displacement y .
- The first boundary condition in (1.2) involves the displacement term y (the boundary condition (1.2) is replaced for instance by $y=0$ or $\dfrac{\partial y}{\partial \upsilon}+y=0$ on $\Gamma_0 \times (0,\infty)$.

The main contribution of this paper is to provide an alternative proof of Lagnese's result (1983) by means of a simple and direct method. The key idea of the proof is to introduce a new energy norm (see Chentouf and Boudellioua (2004) for similar systems and Conrad *et al.* (2002) for one-dimensional wave equation).

2. Preliminaries and Well-posedness of the Problem

In this section, we study the existence and uniqueness of the solutions of the closed-loop system (1.1)-(1.3). Assume, without loss of generality, that for any $x \in \Gamma_1$, $a(x)=a$, where a is a positive constant. Then, consider the state space

$$H=H^1(\Omega)\times L^2(\Omega), \tag{2}$$

equipped with the inner product

$$\begin{aligned}\left\langle (y,z)(\tilde{y},\tilde{z})\right\rangle_H &= \int_\Omega \left(\nabla y \nabla \tilde{y} + z\tilde{z}\right) dx \\ &\quad + \varepsilon\left(\int_\Omega z \; dx + \int_{\Gamma_1} ay \; d\sigma\right)\left(\int_\Omega \tilde{z} dx + \int_{\Gamma_1} a\tilde{y} \; d\sigma\right),\end{aligned} \tag{2.1}$$

where $\varepsilon > 0$ is a constant to be determined.
The first result is stated in the following proposition:

Proposition 1. *The state space $H = H^1(\Omega) \times L^2(\Omega)$ endowed with the inner product (2.1) is a Hilbert space provided that ε is small enough.*

Proof of Proposition 1. It suffices to show that the norm $\|\cdot\|_H$ induced by the inner product (2.1) is equivalent to the usual one $\|\cdot\|_{H^1(\Omega)\times L^2(\Omega)}$, that is, the existence of two positive constants K and $\tilde{K}$ such that

$$K\left\|(y,z)\right\|_{H^1(\Omega)\times L^2(\Omega)} \le \left\|(y,z)\right\|_H \le \tilde{K}\left\|(y,z)\right\|_{H^1(\Omega)\times L^2(\Omega)}. \tag{2.2}$$

Applying Holder's inequality and using trace Theorem Adams (1976) (see also Mikhaïlov (1980)) yield:

$$\begin{aligned}\left\|(y,z)\right\|_H^2 &\le \int_\Omega \left(|\nabla y|^2 + \left[1 + 2\varepsilon\, mes(\Omega)\right]|z|^2\right) dx \\ &\quad + 2\varepsilon a^2 C_1 mes(\Gamma_1)\left(\int_\Omega \left(|\nabla y|^2 + (1+2\varepsilon)|y|^2\right) dx\right),\end{aligned}$$

where C_1 is a positive constant depending on Ω (see Adams (1976) or Mikhaïlov (1980)). Therefore (2.2) holds for a positive constant $\tilde{K}$ depending on $\varepsilon, a, mes(\Gamma_1)$ and $mes(\Omega)$.
For the reverse inequality, we proceed as follows:

$$\begin{aligned}\left\|(y,z)\right\|_H^2 &= \int_\Omega \left(|\nabla y|^2 + |z|^2\right) dx + \varepsilon\left(\int_\Omega z \; dx\right)^2 + \varepsilon\left(\int_{\Gamma_1} ay \; d\sigma\right)^2 \\ &\quad + 2\varepsilon\left(\int_{\Gamma_1} ay \, d\sigma\right)\left(\int_\Omega z \, dx\right).\end{aligned} \tag{2.3}$$

Obviously for any $\delta > 0$, , Young's inequality yields

$$2\left(\int_{\Gamma_1} ay \, d\sigma\right)\left(\int_\Omega z \, dx\right) \ge -a\delta^{-1}\left(\int_\Omega z \, dx\right)^2 - a\delta\left(\int_{\Gamma_1} y \, d\sigma\right)^2. \tag{2.4}$$

Combining (2.4) and (2.3), we get

$$\left\|(y,z)\right\|_H^2 \ge \int_\Omega \left(|\nabla y|^2 + |z|^2\right) dx + \varepsilon\left(1 - a\delta^{-1}\right)\left(\int_\Omega z \, dx\right)^2 + \varepsilon a(a-\delta)\left(\int_{\Gamma_1} y \, d\sigma\right)^2. \tag{2.5}$$

Furthermore, using a classical compactness argument, one can show the following generalized Poincaré inequality:

$$\int_\Omega |y|^2 dx \le C_2\left\{\int_\Omega |\nabla y|^2 dx + \left(\int_{\Gamma_1} y \, d\sigma\right)^2\right\},$$

where $C_2 > 0$ depends on Ω. This, together with (2.5), implies that

$$\begin{aligned}\|(y,z)\|_H^2 \geq & \int_\Omega \left\{ \left[\varepsilon a(a-\delta) C_2^{-1}\right] |y|^2 + \left[1-\varepsilon a(a-\delta)\right] |\nabla y|^2 + |z|^2 \right\} dx \\ & + \varepsilon(\delta - a)\delta^{-1} \left(\int_\Omega z \, dx \right)^2 ,\end{aligned} \tag{2.6}$$

for any $\delta < a$. Finally, applying again Holder's inequality to the last term in (2.6), one can show the existence of a positive constant K depending on a, δ and $mes(\Omega)$ such that the second inequality in (2.2) holds, provided that $\delta < a$ and ε satisfies the following condition

$$\varepsilon < \min\left(\frac{1}{a(a-\delta)}, \frac{\delta}{(a-\delta) mes(\Omega)} \right).$$

This concludes the proof of Proposition 1. ■

We turn now to the formulation of the closed-loop system (1.1)-(1.3) in an abstract form on H. Let $z(\cdot,t) = y_t(\cdot,t)$ and $\Phi(t) = (y(\cdot,t), z(\cdot,t))$. Then, the closed loop system can be written as follows

$$\begin{cases} \Phi_t(t) = A\Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, z_0), \end{cases} \tag{2.7}$$

where A is an unbounded linear operator defined by

$$D(A) = \left\{ \begin{array}{l} (y,z) \in H^1(\Omega) \times H^1(\Omega);\ \Delta y \in L^2(\Omega); \\ \quad \dfrac{\partial y}{\partial \upsilon} = 0 \text{ on } \Gamma_0;\ \dfrac{\partial y}{\partial \upsilon} + az = 0 \text{ on } \Gamma_1 \end{array} \right\}, \tag{2.8}$$

and for any $(y,z) \in D(A)$,

$$A(y,z) = (z, \Delta y). \tag{2.9}$$

It is a simple task to check that the operator $-A$, defined by (2.8)-(2.9), is maximal monotone. Therefore, it follows from semigroups theory Pazy (1983) (see also Brezis (1992)) that:

Lemma 1. (i) *The linear operator A generates a C_0 semigroup of contractions $S(t)$ on $H = \overline{D(A)}$.*

(ii) *For any initial data $\Phi_0 = (y_0, z_0) \in D(A)$, the system (2.7) has a unique strong solution $\Phi(t) = (y(\cdot,t), y_t(\cdot,t)) = S(t)\Phi_0 \in D(A)$ for all $t \geq 0$ such that $\Phi(\cdot) = (y, y_t) \in C^1(\mathbb{R}^+; H) \cap C(\mathbb{R}^+; D(A))$. Moreover, the function $t \mapsto \|A\Phi(t)\|_H$ is decreasing.*

(iii) *For any initial data $\Phi_0 = (y_0, z_0) \in H$, the system (2.7) has a unique weak solution $\Phi(t) = (y(\cdot,t), y_t(\cdot,t)) = S(t)\Phi_0 \in H$ such that $\Phi(\cdot) = (y, y_t) \in C^0(\mathbb{R}^+; H)$.*

3. Asymptotic Behavior

In this section, we will show an asymptotic behavior result for the unique solution of (2.7) in the state space $H = H^1(\Omega)\times L^2(\Omega)$. To do so, we shall first show the following lemma:

Lemma 2. *The resolvent operator* $(\lambda I - A)^{-1} : H \to H$ *is compact for any* $\lambda > 0$ *and hence the canonical embedding* $i : D(A) \to H$ *is compact, where is equipped with the graph norm.*

Proof of Lemma 2. Let $(f, g,) \in H$. We seek $(y, z) \in D(A)$ such that $(I - A)(y, z) = (f, g)$. Equivalently,

$$\begin{cases} z = y - f, & \\ \Delta y - y = -(f + g), & \\ \dfrac{\partial y}{\partial \upsilon} = 0, & \text{on } \Gamma_0, \\ \dfrac{\partial y}{\partial \upsilon} + ay = af, & \text{on } \Gamma_1, \\ y \in H^1(\Omega);\ \Delta y \in L^2(\Omega). & \end{cases}$$

Using Lax-Milgram Theorem (see Brezis (1992)), one can readily show that the above system has a unique solution and thus the operator $(I - A)^{-1}$ exists and maps H into $D(A)$. Finally, by using Sobolev embedding, we deduce that $(I - A)^{-1}$ is compact. The proof of Lemma 2 follows then from the well-known result of Kato (1976). ■

The main result of this paper is:

Theorem 1 . *For any initial data,* $\Phi_0 = (y_0, z_0) \in H$, *the solution* $\Phi(t) = (y(\cdot,t), y_t(\cdot,t))$ *of* (2.7) *tends in* H *to* $(C, 0)$ *as* $t \to +\infty$, *where*

$$C = \{a\, mes(\Gamma_1)\}^{-1}\left(\int_\Omega z_0\, dx + \int_{\Gamma_1} ay_0\, d\sigma\right).$$

Proof of Theorem 1. By a standard argument of density of $D(A^2)$ in H and the contraction of the semigroup $S(t)$, it suffices to prove Theorem 1 for smooth initial data $\Phi_0 = (y_0, z_0) \in D(A^2)$. Let $\Phi(t) = (y(\cdot,t), y_t(\cdot,t)) = S(t)\Phi_0$ be the solution of (2.7). It follows from (ii) of Lemma 1 that the trajectory of solution $\{\Phi(t)\}_{t\ge 0}$ is a bounded set for the graph norm and thus precompact in H by virtue of Lemma 2. Applying LaSalle's principle, we deduce that $\omega(\Phi_0)$ is non-empty, compact, invariant under the semigroup $S(t)$ and $S(t)\Phi_0 \to \omega(\Phi_0)$ as $t \to \infty$ $[10]$. In order to prove the strong stability, it suffices to show that $\omega(\Phi_0)$ reduces to $(C,0)$. To this end, let $\tilde{\Phi}_0 = (\tilde{y}_0, \tilde{z}_0) \in \omega(\Phi_0) \subset D(A)$ and $\tilde{\Phi}(t) = (\tilde{y}(\cdot,t), \tilde{y}_t(\cdot,t)) = S(t)\tilde{\Phi}_0 \in D(A)$ the unique strong solution of (2.7). Recall that it is well-known that $\left\|\tilde{\Phi}(t)\right\|_H$ is constant Haraux (1991) and thus $\frac{d}{dt}\left(\left\|\tilde{\Phi}(t)\right\|_H^2\right) = 0,$ i.e.,

$$< A\tilde{\Phi}, \tilde{\Phi} >_H = -\int_{\Gamma_1} a|z|^2 d\sigma = 0. \tag{3.1}$$

This implies that $\tilde{z} = \tilde{y}_t = 0$ on Γ_1 and therefore $\tilde{y}$ is solution of the system
System:

$$\begin{cases} \tilde{y}_{tt} - \Delta\tilde{y} = 0, & \\ \dfrac{\partial y}{\partial \upsilon} = 0, & \text{on } \Gamma_0, \\ \tilde{y}_t = \dfrac{\partial \tilde{y}}{\partial \upsilon} = 0, & \text{on } \Gamma_1, \\ \tilde{y}(0) = \tilde{y}_0;\ \tilde{y}_t = \tilde{z}_0, & \\ \tilde{y} \in H^1(\Omega);\ \Delta\tilde{y} \in L^2(\Omega), & \end{cases} \tag{3.2}$$

whereas $\tilde{z} = \tilde{y}_t$ is solution of the system

$$\begin{cases} \tilde{z}_{tt} - \Delta\tilde{z} = 0, & \\ \dfrac{\partial \tilde{z}}{\partial \upsilon} = 0, & \text{on } \Gamma_0, \\ \tilde{z} = \dfrac{\partial \tilde{z}}{\partial \upsilon} = 0, & \text{on } \Gamma_1, \\ \tilde{z}(0) = \tilde{z}_0;\ \tilde{z}_t(0) = \Delta\tilde{y}_0. & \end{cases} \tag{3.3}$$

Obviously, to deduce the desired result, it suffices to show that $\tilde{y} = constant$ is the only solution of (3.2). To do so, we first use the standard Holmgren's uniqueness theorem (see John (1982)) for the system (3.3) to conclude that $\tilde{z} \equiv 0$. Thus the system (3.2) is reduced to the elliptic problem:

$$\begin{cases} \Delta\tilde{y} = 0, \\ \dfrac{\partial \tilde{y}}{\partial \upsilon} = 0, \quad \text{on } \Gamma, \end{cases}$$

which clearly yields that $\tilde{y} \equiv$ constant. This, together with (3.1), implies the desired result. ■

Remark 1. *Integrating with respect to* x *and* t *and using Green formula for the closed loop system* (1.1)-(1.3), *we obtain the following identity:*

$$\int_{\Omega} y_t \, dx + \int_{\Gamma_1} ay \, d\sigma = \int_{\Omega} z_0 \, dx + \int_{\Gamma_1} ay_0 \, d\sigma.$$

Furthermore, if the initial data (y_0, z_0) *satisfies the additional condition*

$$\int_{\Omega} z_0 \, dx + \int_{\Gamma_1} ay_0 \, d\sigma = 0,$$

then the constant C *of Theorem* 1 *is zero, i.e.,* $\left(y(\cdot,t), y_t(\cdot,t)\right) \to (0,0)$ *in* H *as* $t \to +\infty$.

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