

Zero-Hopf Bifurcation in the Generalized Stretch-Twist-Fold Flow

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ABSTRACT: A zero-Hopf equilibrium in a three-dimensional system is an isolated equilibrium point which has a zero eigenvalue and a simple pair of purely imaginary eigenvalues. In general, for such an equilibrium, there is no theory for finding when some periodic solutions are bifurcated by perturbing the parameters of the system. In this work, we describe the values of the parameters for which a zero-Hopf equilibrium occurs at the equilibrium points in the generalized stretch-twist-fold flow. Thus, only one condition for parameters of the generalized stretch-twist-fold flow introduced in a system (Eq. 1) is found for which the equilibrium point is a zero-Hopf equilibrium. For this condition, we use the averaging method to provide the existence of a periodic solution, which bifurcates from the zero-Hopf equilibrium point. The main result in this paper is Theorem 1, which gives a periodic solution of the generalized stretch-twist-fold flow.

Keywords: Stretch-twist-fold flow; Zero-Hopf equilibrium point; Zero-Hopf bifurcation; Periodic solution; The averaging method.

تشعب هوبف الصفري في إمتداد- إلتواء- التدفق المطوي المعمم

نيازي هادي حسين و ازاد ابراهيم امين

المخلص: إن توازن هوبف الصفري في نظام ثلاثي الأبعاد هو نقطة توازن معزولة قيمتها الذاتية صفر ولها زوج بسيط من القيم الذاتية العقدية البحتة. لا توجد طريقة نظرية، بشكل عام، لتمثل هذا التوازن عندما يتم تشعب بعض الحلول الدورية عن طريق اضطراب معاملات النظام. نوصف، في هذا العمل، قيم المعاملات التي يظهر فيها توازن هوبف الصفري عند نقاط التوازن في تمديد - التواء التدفق المطوي المعمم. وبالتالي، تم ادخال شرط واحد فقط على معاملات تمديد - التواء التدفق المطوي المعمم في نظام (معادلة 1) تم ايجاده بحيث تكون نقطة التوازن هي توازن هوبف الصفري. بالنسبة إلى هذا الشرط، نستخدم طريقة المتوسط لتوفير وجود حل دوري يتفرع من نقطة توازن هوبف الصفري. النتيجة الرئيسية في هذه الورقة هي النظرية 1 التي تقدم حلاً دورياً لتمديد - التواء التدفق المطوي المعمم.

الكلمات المفتاحية: امتداد - التواء - التدفق المطوي، نقطة توازن هوبف الصفري، تشعب هوبف الصفري، الحل الدوري، طريقة المتوسط.

1. Introduction

As a special case of Stokes flows, the stretch-twist-fold (STF) flow normally appears in dynamo theory. This flow was first formulated in the work of Vainshtein and Zel'dovich [1]. On the principle of this research, Moffatt in [2] invented a particular quadratic flow to represent the STF action. There are also very extensive studies of the STF flow; one can see [3-8]. Dependent on the STF flow, a generalized stretch-twist-fold (GSTF) flow was suggested in [9] by Jianghong and Qigui. They added two extra parameters in the STF flow in order to further investigate the STF system. The parameters for which the Hopf equilibrium point occurs in the GSTF flow are described in [9]. The new parameter not only preserves the original dynamic properties but also has some novel characteristics.



Now, in the following, the GSTF system is considered

$$\left. \begin{aligned} \dot{x} &= a_1 z - 8(b+1)xy, \\ \dot{y} &= 11x^2 + 3y^2 + z^2 + a_2 xz - 3c^2, \\ \dot{z} &= -a_1 x + 2yz - a_2 xy, \end{aligned} \right\} \quad (1)$$

where a_1 and a_2 are real positive parameters and are associated with the ratios of intensities of the SFT ingredients of the flow, and b, c are real parameters added in the original STF flow which determines the bifurcations of the system. Moreover, an over-dot indicates the derivative with respect to independent variable t . System (1) remains non-conservative anymore ($\nabla V = -8b y$), and the original STF flow does not exhibit the Hopf bifurcation, but the authors in [1] showed that system (1) has the Hopf bifurcation, which is very interesting. They used the normal form theory for studying Hopf bifurcations in detail. The authors in [9] proved that system (1) has no Darboux first integral when $a_1 \neq 0, b = 0$ and $c = \pm 1$.

In this work, our main goal is to study the existence of the zero-Hopf equilibria; we put a suitable condition on the singular point to obtain a zero-Hopf equilibrium point, and which gives the zero-Hopf bifurcations in the GSTF flow (1). This is described in Proposition 1. Generally, for this kind of equilibrium there is no theory to explain when it bifurcates some periodic solutions which move the parameters of the system. We then introduce a suitable rescaling of variables which transform the GSTF system (1) into the normal form system; the condition for the existence of a periodic solution which bifurcates from the zero-Hopf equilibrium is given by using the averaging method. This method will be given in Section 4 also contains some basic results to obtain a periodic solution of three dimensional polynomial differential systems. This investigation provides a new results for a periodic solution of the GSTF flow, which we have not found in the literature. Whether or not a system has periodic solutions is very interesting and important in non-linear differential systems. The existence of periodic solutions for this kind of bifurcation in a 3D differential system has been widely addressed in [10-17].

Actually, in [1] the authors investigate the Hopf bifurcation of system (1). Their results are more general than the present ones, but our results are more accurate for finding periodic solution produced from a zero-Hopf equilibrium.

The reason is that we follow the conditions of the averaging method which is described in Theorem 2. These conditions clearly guarantee that there is a periodic solution of the given system.

When the GSTF system (1) possessing a zero-Hopf equilibrium point at the origin of coordinates possess a zero-Hopf bifurcation, it produces some periodic solutions. For performing this, we consider ϵ -perturbations of the values of the parameters for which new ones have a zero-Hopf equilibrium. The small parameter ϵ is necessary to be used in the averaging method, and the analysis of a zero-Hopf bifurcation will be executed with respect to it.

This work is arranged as follows. In Section 2, the equilibrium of system (1) is investigated. In Section 3, the explicit parametric condition of existence of a zero-Hopf equilibrium and a periodic solution of system (1) is given. Section 4 explains the averaging method of the first order. In Section 5, we give the proofs of the results presented in Section 3, and we also describe the stability or instability of the periodic solution which is produced in such a zero-Hopf bifurcation of the GSTF system (1). The last section concludes the work.

2. Equilibrium

The equilibria of system (1) for $a_1 \neq 0$ are all on the following surface of the ellipsoid

$$\left(1 - \frac{8}{3}b\right)x^2 + y^2 + z^2 = c^2. \quad (2)$$

We obtain (2) by solving the following three equations

$$a_1 z - 8(b+1)xy = 0, \quad 11x^2 + 3y^2 + z^2 + a_2 xz - 3c^2 = 0, \quad -a_1 x + 2yz - a_2 xy = 0.$$

For $a_1 = 0$, the equilibria are the solutions of the following equations

$$8(b+1)xy = 0, \quad 11x^2 + 3y^2 + z^2 + a_2 xz - 3c^2 = 0, \quad 2yz - a_2 xy = 0.$$

If $y = 0$, the equilibria are a quadratic curve

$$11x^2 + z^2 + a_2 xz - 3c^2 = 0, \quad y = 0,$$

If $y \neq 0$, then the two isolated points $E_+(0, c, 0)$ and $E_-(0, -c, 0)$ are the equilibria. We are only interested in the case where $a_1 \neq 0$ and $c = 0$. According to equation (2), these give us the origin of coordinates as the only equilibrium point of system (1).

3. Statement of the main results

If an isolated equilibrium point of a 3D autonomous differential system has a zero eigenvalue and a pair of purely imaginary eigenvalues then this kind of equilibrium point is called the zero-Hopf equilibrium (also called a saddle-node Hopf bifurcation). It comes out that its unfolding owns the rich dynamics in a small neighborhood of the origin. This means that, unfolding is used to describe the dynamics in small neighborhoods of the equilibria and their perturbations (see for instance, Guckenheimer [18], Guckenheimer and Holmes [19], Scheurle and Marsden [20], Kuznetsov [21], and the references therein). In some cases a zero-Hopf bifurcation may lead to a local birth of “chaos” as can be found in the work of Baldomá and Seara [22], and Broer and Vegter [23].

Our goal in the next proposition is to find a condition in which a zero-Hopf equilibrium point takes the place of system (1).

Proposition 1. There is only a one-parameter family of system (1) in which the equilibrium point localized at the origin of coordinates is a zero-Hopf equilibrium point. That is

$$c = 0. \tag{3}$$

We will prove this proposition in Section 4. This condition is the only one-parameter family of system (1) which exhibits a zero-Hopf equilibrium localized at the equilibrium point.

The next theorem tells us there is a small amplitude periodic solution which bifurcates from a zero-Hopf equilibrium point of system (1).

Theorem 1. Let the vector given by $(a_1, a_2, b, c) = (\tilde{a}_1 + \epsilon \alpha_1, \tilde{a}_2 + \epsilon \alpha_2, \tilde{b} + \epsilon \alpha_3, \epsilon \alpha_4)$ with $\epsilon > 0$ a sufficiently small parameter. If

$$\alpha_4 \neq 0, \text{ and } \tilde{b} \neq \frac{3}{4}.$$

then system (1) has a zero-Hopf bifurcation at the equilibrium point which localizes at the origin of coordinates, and when $\epsilon = 0$, a limit cycle (periodic solution) is produced at this equilibrium. Furthermore, a stability or instability of this limit cycle is the same as the equilibrium point of a planar differential system with the following eigenvalues

$$\pm \frac{1}{\omega} \sqrt{-6\alpha_4^2 (3 + 4\tilde{b})}.$$

In Section 4, the above theorem will be proved via the averaging method of first order. This method is explained briefly in the following section.

4. Periodic solutions via the averaging method

In this work, we use the averaging method to study a periodic solution of the GSTF flow (1). The averaging method is a classical and developed tool to determine the behavior of the dynamics of non-linear differential systems, and in particular of their periodic orbits. This method has a long history which starts with the works of Lagrange and Laplace. An intuitive justification of the process was provided by them. The first formalization of this method is due to Fatou [24]. Bogoliubov and Krylov [25], and Bogoliubov [26], made important practical and theoretical contributions using this method.

In the following, the basic results concerning the averaging theory of the first order can be summarized as follows

Theorem 2. Consider the following two initial value problems:

$$\dot{x} = \epsilon f_1(t, x) + \epsilon^2 f_2(t, x, \epsilon), \quad x(0) = x_0, \tag{4}$$

and

$$\dot{y} = \epsilon g(y), \quad y(0) = x_0, \tag{5}$$

where $x, y, x_0 \in \Omega$ is an open subset of \mathbb{R}^n , with $t \in [0, \infty)$. Furthermore, we suppose that $f_1(t, x)$ and $f_2(t, x, \epsilon)$ are periodic of T-period in the variable t , and $g(y)$ is the averaged function of $f_1(t, y) \in \Omega$. We mean that

$$g(y) = \frac{1}{T} \int_0^T f_1(t, y) dt. \tag{6}$$

Suppose

1. $f_1, \frac{\partial f_1}{\partial x}, \frac{\partial^2 f_1}{\partial x^2}, f_2$ and $\frac{\partial f_2}{\partial x}$ are all defined, continuous in $\mathbb{R}^+ \times \Omega$, and these expressions are bounded by a constant independent of $\epsilon \in (0, \epsilon_0]$.
2. T is a constant independent of ϵ , and
3. $y(t)$ belongs to Ω on the interval of time $[0, 1/\epsilon]$.

Then the following statements hold

- On the time scale for $\frac{1}{\epsilon}$, we have that $x(t) - y(t) = O(\epsilon)$, as $t \rightarrow 0$.
- If p is an equilibrium point of the averaged system (5) such that

$$\det \left(\frac{\partial g}{\partial y} \right)_{y=p} \neq 0. \quad (7)$$

Then, there is a T -periodic solution $\phi(t, \epsilon)$ for system (4) which is close to p and such that $\phi(t, \epsilon)$ approaches to p , as ϵ approaches to zero. Furthermore, the equilibrium point of averaging differential system (5) has the same stability or instability as the periodic solution $\phi(t, \epsilon)$, and the equilibrium point with eigenvalues. Actually, the equilibrium point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t, \epsilon)$. The proof of the averaging method of the first order to determine periodic solutions can be found in the book of Sanders and Verhulst [27], see Theorems 11.5 and 11.6.

5. Proof of the main results

Proof of Proposition 1: For $a_1 \neq 0$ and $c \neq 0$, system (1) has two isolated equilibria $E_{\pm} = (0, \pm c, 0)$, which collide at the origin coordinates where $c = 0$. The proof is made by calculating directly the eigenvalues at the origin coordinates. Note that the linear part of system (1) has the following characteristic polynomial at the origin of coordinates

$$p(\lambda) = \lambda^3 + a_1^2 \lambda = \lambda(\lambda^2 + a_1^2).$$

Putting that $p(\lambda) = \lambda(\lambda^2 + \delta^2) = 0$, where $\delta = a_1 > 0$, thus the proposition follows.

Proof of Theorem 1: If we consider the ϵ -perturbations $(a_1, a_2, b, c) = (\tilde{a}_1 + \epsilon \alpha_1, \tilde{a}_2 + \epsilon \alpha_2, \tilde{b} + \epsilon \alpha_3, \epsilon \alpha_4)$, with $\epsilon > 0$ a sufficiently small parameter, then, system (1) becomes

$$\left. \begin{aligned} \dot{x} &= (\tilde{a}_1 + \epsilon \alpha_1)z - 8(\tilde{b} + \epsilon \alpha_3 + 1)xy, \\ \dot{y} &= 11x^2 + 3y^2 + z^2 + (\tilde{a}_2 + \epsilon \alpha_2)xz - 3\epsilon \alpha_4^2, \\ \dot{z} &= -(\tilde{a}_1 + \epsilon \alpha_1)x + 2yz - (\tilde{a}_2 + \epsilon \alpha_2)xy. \end{aligned} \right\} \quad (8)$$

By the rescaling of variables $(x, y, z) = (\epsilon X, \epsilon Y, \epsilon Z)$, system (8) in the new variables (X, Y, Z) becomes

$$\left. \begin{aligned} \dot{X} &= (\tilde{a}_1 + \epsilon \alpha_1)Z - 8(\epsilon \tilde{b} + \epsilon^2 \alpha_3 + \epsilon)XY, \\ \dot{Y} &= \epsilon Z^2 + \epsilon (\tilde{a}_2 + \epsilon \alpha_2)XZ + \epsilon (11X^2 + 3Y^2 - 3\alpha_4^2), \\ \dot{Z} &= -(\tilde{a}_1 + \epsilon \alpha_1)X + 2\epsilon YZ - (\epsilon \tilde{a}_2 + \epsilon^2 \alpha_2)XY. \end{aligned} \right\} \quad (9)$$

Now, the linear part at the origin of system (9) can be written in its real Jordan normal form, namely when $\epsilon = 0$, we shall transform the linear part at the origin of the differential equation (9) into its real Jordan normal form, which is as

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \omega = \tilde{a}_1 > 0.$$

For doing this we use the linear change of variables $(X, Y, Z) \rightarrow (U, V, W)$ given by

$$(X, Y, Z) = P(U, V, W), \quad \text{where } P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (10)$$

Clearly, P^{-1} exists. Then system (9) can be written in the new variables (U, V, W) in the following

$$\left. \begin{aligned} \dot{U} &= -V \omega + ((-8 \tilde{b} - 8)UW - V \alpha_1)\epsilon - 8 \epsilon^2 U W \alpha_3, \\ \dot{V} &= U \omega + (U \alpha_1 + 2 W V + U W \tilde{a}_2)\epsilon + \epsilon^2 W \alpha_2, \end{aligned} \right\} \quad (11)$$

$$\dot{W} = (11 U^2 + 3 W^2 + V^2 - U V \tilde{a}_2 - 3 \alpha_4^2)\epsilon - \epsilon^2 U V \alpha_2.$$

To study the periodic solutions of system (11) when ϵ is too small, we use the following a class of cylindrical coordinates

$$(U, V, W) = (r \cos(\theta), r \sin(\theta), W), \quad r > 0. \quad (12)$$

This is a topological change of variables in a deleted neighborhood of origin coordinates.

$$\begin{aligned} \dot{r} &= \epsilon (2 (\epsilon \alpha_2 - \tilde{a}_2) \sin(\theta) \cos(\theta) + (8 \epsilon \alpha_3 + 8 \tilde{b} + 10) \cos^2(\theta)) r W, \\ \dot{\theta} &= \omega + \epsilon \alpha_1 + \epsilon ((10 + 8 \epsilon \alpha_3 + 8 \tilde{b}) \sin(\theta) \cos(\theta) + (\tilde{a}_2 + \epsilon \alpha_2) \cos^2(\theta)) W, \\ \dot{W} &= -3 \epsilon \alpha_4^2 + \epsilon (10 \cos^2(\theta) - (\tilde{a}_2 + \epsilon \alpha) \cos(\theta) \sin(\theta)) r^2 + 3 \epsilon W^2, \end{aligned}$$

where $\omega = \tilde{a}_1 > 0$. This implies that there exists a small neighborhood of $(r, W) = (0, 0)$ such that $\dot{\theta} \neq 0$ for all θ in this neighborhood. Furthermore, we consider an equivalent system of (12) in the neighborhood of $(r, W) = (0, 0)$, introducing also θ as the new independent variable. Thus, we obtain the following system

$$\left. \begin{aligned} \frac{dr}{d\theta} &= \epsilon f_{1,1} + \frac{\epsilon^2}{\omega^2} f_{2,1}(r, \theta, W) + O(\epsilon^3), \\ \frac{dW}{d\theta} &= \epsilon f_{1,2} + \epsilon^2 f_{2,2}(r, \theta, W) + O(\epsilon^3), \end{aligned} \right\} \quad (13)$$

where

$$\begin{aligned} f_{1,1} &= -r \frac{W}{\omega} \left(8(\cos(\theta))^2 \tilde{b} + 10(\cos(\theta))^2 - 2 - \sin(\theta) \cos(\theta) \tilde{a}_2 \right), \\ f_{1,2} &= \frac{1}{\omega} \left(-r^2 \cos(\theta) \sin(\theta) \tilde{a}_2 + 3W^2 - 3\alpha_4^2 + r^2 + 10r^2 (\cos(\theta))^2 \right), \\ f_{2,1} &= \left(8(\omega \alpha_3 - \tilde{b} \alpha_1) \cos^2(\theta) + \sin(\theta) \cos(\theta) \tilde{a}_2 \alpha_1 + 2\alpha_1 - \cos(\theta) \sin(\theta) \alpha_2 \omega - 10 \cos^2(\theta) \alpha_1 \right) r W \\ &\quad + \left(-160 \cos^3(\theta) \sin(\theta) \tilde{b} + 20 \sin(\theta) \cos(\theta) + 12 \cos^2(\theta) \tilde{a}_2 + \cos^3(\theta) \sin(\theta) \tilde{a}_2^2 \right. \\ &\quad + 8 \cos^2(\theta) \tilde{a}_2 \tilde{b} - 64 \cos^3(\theta) \tilde{b}^2 \sin(\theta) - 16 \cos^4(\theta) \tilde{a}_2 \tilde{b} - 20 \cos^4(\theta) \tilde{a}_2 \\ &\quad \left. - 100 \cos^3(\theta) \sin(\theta) + 16 \sin(\theta) \cos(\theta) \tilde{b} \right) W^2 r, \\ f_{2,2} &= 3\alpha_1 \alpha_4^2 + \left(3\alpha_4^2 \cos^2(\theta) \tilde{a}_2 + 30\alpha_4^2 \sin(\theta) \cos(\theta) + 24\alpha_4^2 \sin(\theta) \cos(\theta) \tilde{a} \right) W \\ &\quad + \left(-\alpha_1 + \sin(\theta) \cos(\theta) \tilde{a}_2 \alpha_1 - 10 \cos^2(\theta) \alpha_1 - \cos(\theta) \sin(\theta) \alpha_2 \omega \right) r^2 - 3W^2 \alpha_1 \\ &\quad + \left(\cos^3(\theta) \sin(\theta) \tilde{a}_2^2 - 8 \cos^4(\theta) \tilde{a}_2 \tilde{b} - 8 \sin(\theta) \cos(\theta) \tilde{b} - 80 \cos^3(\theta) \sin(\theta) \tilde{b} \right. \\ &\quad \left. - 100 \cos^3(\theta) \sin(\theta) + 9 \cos^2(\theta) \tilde{a}_2 - 20 \cos^4(\theta) \tilde{a}_2 + 8 \cos^2(\theta) \tilde{a}_2 \tilde{b} \right. \\ &\quad \left. - 10 \sin(\theta) \cos(\theta) \right) W r^2 + \left(-30 \sin(\theta) \cos(\theta) - 24 \sin(\theta) \cos(\theta) \tilde{b} - 3(\cos(\theta))^2 \tilde{a}_2 \right) W^3. \end{aligned}$$

Hence, system (13) is transformed into the normal form with respect to the parameter ϵ in order to apply the averaging method. We now consider the averaging differential system of first order for system (13). Using Theorem 2, we note that all the assumptions of Theorem 2 are satisfied for the differential system (13), where we put $t = \theta$, $T = 2\pi$, $x = (r, W) \in (0, \infty) \times \mathbb{R}$, $x_0 = (r_0, W_0)$ and

$$f_1(t, x) = f_1(\theta, r, W) = \begin{bmatrix} f_{1,1}(\theta, r, W) \\ f_{1,2}(\theta, r, W) \end{bmatrix}$$

Now we denote $g(r, W) = (g_{11}(r, W), g_{12}(r, W))$ and compute the integrals (6), i.e. the averaged functions with $y = (r, W)^T$,

$$g(r, W) = \begin{bmatrix} g_{1,1}(r, W) \\ g_{1,2}(r, W) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} f_{1,1} d\theta \\ \frac{1}{2\pi} \int_0^{2\pi} f_{1,2} d\theta \end{bmatrix} = \begin{bmatrix} -\frac{rW}{\omega} (3 + 4\tilde{b}), \\ \frac{3}{\omega} (2r^2 - \alpha_4^2 + W^2) \end{bmatrix}. \quad (14)$$

Therefore, the $g_{11}(r, W), = g_{12}(r, W) = 0$ has a unique real solution (r^*, W^*) where

$$(r^*, W^*) = \left(\frac{1}{\sqrt{2}} \sqrt{\alpha_4^2}, 0 \right) \neq 0.$$

In addition, to compute the Jacobian we have

$$\frac{\partial g(r, W)}{\partial(r, W)} = \begin{bmatrix} \frac{\partial g_{1,1}}{\partial r} & \frac{\partial g_{1,1}}{\partial W} \\ \frac{\partial g_{1,2}}{\partial r} & \frac{\partial g_{1,2}}{\partial W} \end{bmatrix} = \begin{bmatrix} -\frac{W}{\omega} (3 + 4\tilde{b}) & -\frac{r}{\omega} (3 + 4\tilde{b}) \\ \frac{12r}{\omega} & \frac{6W}{\omega} \end{bmatrix},$$

That is, the Jacobian matrix of (14) at the point (r^*, W^*) takes the value

$$\left| \frac{\partial g(r, W)}{\partial(r, W)} \right|_{\left(\frac{1}{\sqrt{2}} \sqrt{\alpha_4^2}, 0 \right)} = \frac{6\alpha_4^2 (3 + 4\tilde{b})}{\omega^2},$$

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which is different from zero where $\tilde{b} \neq -\frac{3}{4}$ and $\alpha_4 \neq 0$, and the eigenvalues at the solution (r^*, W^*) are

$$\pm \frac{1}{\omega} \sqrt{-6\alpha_4^2 (3 + 4\tilde{b})}$$

These eigenvalues are the same as given in (3). This shows that when

$$\tilde{b} < -\frac{3}{4}, \tag{15}$$

the periodic solution is unstable, and it has a stable manifold and an unstable manifold both formed by two cylinders.

From the averaging theorem (Theorem 2), the proof is deduced once we show that a periodic solution corresponding to (r^*, W^*) produces a periodic solution which bifurcates from the origin of system (8) when $\epsilon = 0$.

Theorem 2 guarantees that, for $\epsilon > 0$ sufficiently small, there is a periodic solution corresponding to the point (r^*, W^*) of the form $(r(\theta, \epsilon), W(\theta, \epsilon))$ such that $(r(0, \epsilon), W(0, \epsilon)) \rightarrow (r^*, W^*)$ when ϵ approaches to zero. So, the differential system (11) possess a periodic solution

$$(U(\theta, \epsilon), V(\theta, \epsilon), W(\theta, \epsilon)) = (r\cos(\theta), r\sin(\theta), W), \tag{16}$$

for $\epsilon > 0$ sufficiently small. Consequently, under the change of variables (10) the differential system (9) possesses the periodic solution $(X(\theta), Y(\theta), Z(\theta))$ obtained from the relation (16). Finally, for $\epsilon > 0$ sufficiently small, system (8) possesses the periodic solution

$$(x(\theta), y(\theta), z(\theta)) = (\epsilon X(\theta), \epsilon Y(\theta), \epsilon Z(\theta))$$

tending to the origin of coordinates when ϵ approaches to zero. Then, it is the periodic solution which starts at the zero-Hopf equilibrium point located at the origin when ϵ is equal to zero. This completes the proof.

Conclusion

In this article, we have considered the Generalized Stretch-Twist-Fold Flow (GSTF) system flow originally suggested by Jianghong and Qigui. They added two extra parameters in the STF flow in order to further investigate the STF system. We have then described the values of the parameters for which a zero-Hopf equilibrium occurs at the origin for this system. We showed that there is only one condition for parameters of this type of system (Eq. 1) for which the equilibrium point is a zero-Hopf equilibrium. Moreover, we have used the averaging method of first order (see Theorem 2) to determine a periodic solution, which bifurcates from the zero-Hopf equilibrium point, see Theorem 1. This theorem gives a periodic solution of the GSTF. It follows from theorem 2 that any sufficiently small system has a periodic solution of period near 2π . Clearly this periodic orbit tends to the origin of coordinates when tends to zero. Therefore, this is the small amplitude periodic solution, which starts at the zero-Hopf equilibrium.

Conflict of interest

The authors declare no conflict of interest.

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