SQU Journal For Science, 14 (2009) 45-51

Stability of The Synchronization Manifold in An All-To-All Time LAG-Diffusively Coupled Oscillators

Adu A.M. Wasike

Department of Mathematics, Masinde Muliro University of Science and Technology, P.O.Box 190 Kakamega, Kenya. Email: aduwasike@yahoo.com.

ثبات متنوعات التوافق في المذبب الشبكي المزدوج ذو الإعاقة الزمنية من الكل إلى الكل

أدو أ.م. واسك

خلاصة: في هذه الورقة درسنا أنظمة شبكات المذبذبات المتطابقة والمترابطة كل واحد مع الأخر بشبكة ترابط ذات اعاقة زمنية. ولتقدير زمن الإعاقة الذي تتبعه ثبوتية النظام استعملنا الانقسام الطبيعي للنظام إلى متنوع توافق ومتنوع توافق ومتنوع مستعرض. كما لاحظنا ان كل مذبذب يتبع حل دوري وحيد.

ABSTRACT: We consider a lattice system of identical oscillators that are all coupled to one another with a diffusive coupling that has a time lag. We use the natural splitting of the system into synchronized manifold and transversal manifold to estimate the value of the time lag for which the stability of the system follows from that without a time lag. Each oscillator has a unique periodic solution that is attracting.

KEYWORDS: Exponential stability, Time lag, Gronwall's Inequality.

1. Introduction

In the past few years, there have been many papers concerned with the synchronization and stability of systems with diffusive coupling (see for instance Afraimovich *et. al.*, 1983, 1986, Chow and Liu 1977, Hale 1977, Wasike 2003, 2007, and references therein). Other than numerical estimates have been studied by Rossoni *et al.*, 2005). There are few results that deal with the problem of stability of the synchronization manifold in a system with a time lag in the coupling in a mathematically rigorous manner (see for instance Wasike 2003, 2007).

Let us suppose that we have n subsystems $z_j \in \mathrm{R}^d$, $j = 1, 2, ..., n,$ with the dynamics of each z_j given by the solutions of the system of d first order equations

$$\dot{z}_j = g\left(z_j\right), \qquad g \in c\left(U, \mathrm{R}^d\right), \tag{1}$$

where the dot denotes differentiation with respect to time t and U is some open set in $\mathbb{R}^d$. Suppose that, for each j, there is a compact global attractor for equation (1); that is, there is a compact set which is invariant under the flow defined by equation (1) and that the $w-$limit set of each orbit of equation (1) belongs to this set.

Now suppose that these systems are coupled linearly with terms that involve some constant time lag, $r>0$, to obtain

$$\dot{z}(t)=kB\big(z(t),z(t-r)\big)+f\big(z(t)\big), \tag{2}$$

where $k>0$, is a positive constant representing the coupling strength, $z(t)=\big(z_1(t),...,z_n(t)\big)$, $f\big(z(t)\big)=\big(g\big(z_1(t)\big),...g\big(z_n(t)\big)\big)$ and $B\big(z(t),z(t-r)\big)$ is a linear function in $z(t)$ and $z(t-r)$ indicating the coupling configuration. As a specific example, we shall be interested in the coupling configuration given by

$$B\big(z(t),z(t-r)\big)=-(n-1)I_n\otimes I_d z(t)+\breve{I}_n\otimes I_d z(t-r), \tag{3}$$

where I_n, I_d are identity matrices of order n and d respectively, $\breve{I}_n$ is an $n\times n$ matrix whose entries are all 1's except the principal diagonal that is all zeroes, $\otimes$ is the Kronecker product (see for instance Graham 1981). Indeed Equation (2) can be written

$$\dot{z}_j(t)=k\left(-(n-1)z_j(t)+\sum_{i=1,\ i\neq j}^{n} z_i(t-r)\right)+g\big(z_j(t)\big) \tag{4}$$

This type of coupling corresponds to an all-to-all nearest neighbour diffusive coupling.

2. Preliminaries

Let $X=C\big([-r,0],R^{nd}\big)$ be the space of continuous functions from $[-r,0]$ to R^{nd} endowed with the usual supremum topology. For any $\varphi\in X$, Equation (2) has a unique solution $z(t,\varphi)$ with $z(0,\varphi)(\theta)=\varphi(\theta)$ for θ on the interval $[-r,0]$. If we let $\big(T(t)\varphi\big)(\theta)=z(t+\theta,\varphi), \theta\in[-r,0]$ and assume that all solutions are uniquely defined for $t\geq 0,$ then $T(t)$ is C^0-semigroup on X. Assume that system (4) defines a semiflow $T_k(t)$ and has a global attractor A_k; that is, A_k is a compact set which is invariant $\big(T_k(t)A_k=A_k,\quad t\geq 0\big)$ and, for any bounded set $B\subset X$, $dist\,x\big(T_k(t)B,A_k\big)\to 0$ as $t\to\infty$. The global attractor A_k is uniformly bounded with respect to k. Let A_∞ be the global attractor for the equation

$$\dot{z}(t)=f\big(z(t)\big) \tag{5}$$

With this notation, we say that system (4) is *synchronized* if the global attractor A_k belongs to the diagonal set

$$M=\big\{z(t,\varphi)=\big(z_1(t,\varphi),...z_n(t,\varphi)\big)\,|\,z_1(t,\varphi)=...=z_n(t,\varphi)\big\},$$

that is invariant under $T(t)$. M will be an inertial manifold, see for instance Hale (1997).
From the definition of the attractor, this implies that

$$dist\, x\left(z(t,\varphi), M\right) \to 0, \text{ as } t \to \infty \text{ for all } \varphi \in A_k;$$

that is, the difference $z_j(t) - z_i(t) \to 0$ as $t \to \infty$ for all i, j. Of course, for the system to be synchronized we have to show that $A_k = A_\infty$ for k sufficiently large. This is the case if we consider identical subsystems. In this case the linear operator $kB\left(z(t), z(t-r)\right)$ has zero as an eigenvalues lie to left-half of the complex plane. This is why we make the following hypotheses on $kB\left(z(t), z(t-r)\right)$.

H$_1$ for each $k, 0$ is an eigenvalue $kB\left(z(t), z(t-r)\right)$ and $e = Vec(1,1,\ldots,1) \in \mathrm{R}^{nd}$ is its corresponding generalized eigenvector. Moreover, all other eigenvalues of $kB\left(z(t), z(t-r)\right)$ lie to left-half of the complex plane for $0 < r < \infty$.

H$_2$ $T_k(t)$ is dissipative and has a compact global attractor A_k which is invariant under $T_k(t)$. Furthermore $dist\left(T_k(t) A_k,\ M\right) \to 0$ as $t \to \infty$ and $k > 0$ for all $0 < r < \infty$.

H$_3$ Assume the semigroup can be represented in a natural way as a linear part $DT_k(t)$ plus a nonlinear part and $DT_k(t) M \subset M$ for all $t \geq 0$.

The verification of the stability of the manifold M proceeds by going through the following steps:

(1) We introduce a new coordinate $X = M \oplus M^\perp$, where $M^\perp$ is also invariant under the linear flow $DT_k(t)$,

(2) Show that there exists $L \leq 1, c_k > 0$ such that

$$\left\| DT_k(t) \mid M^\perp \right\| \leq le^{-ck^t},\ t \geq 0$$

This is the scheme that is typically followed in many applications (see for instance Hale, (1996).
Let us check that the matrix of the coupling configuration $kB\left(z(t), z(t-r)\right)$ satisfies **H$_1$**. The eigenvalues λ, of the linear operator $kB\left(z(t), z(t-r)\right)$ are given by the zeroes of

$$\Delta(\lambda) = \left(\left(\Delta_s(\lambda)\right)^{n-1} \Delta_c(\lambda)\right)^d, \tag{6}$$

where

$$\Delta_c(\lambda) = \lambda + k(n-1)\left(1 - e^{-\lambda r}\right), \tag{7}$$

and

$$\Delta_s(\lambda) = \lambda + (n-1)k + ke^{-\lambda r} \tag{8}$$

Clearly $\lambda = 0$ is an eigenvalue of $\Delta_c(\lambda) = 0$ and the corresponding generalized eigenvector is $e = Vec(1,1,...,1) \in \mathbb{R}^{nd}$. As long as $k > 0,\ n \geq 2,$ other than 0, all roots of $\Delta(\lambda)$ have negative real parts for all $r \geq 0,$ (see Bose (1989), Theorem 3.1 page 143).

From $\mathbf{H_1}$ we can introduce a new coordinate system $z = ye + \tilde{e}w,\ \ w = (w_1, w_2, ..., w_{n-1})^T,\ \ w \in \mathbb{R}^{nd-d},$ $y \in \mathbb{R}^d,$

$$\begin{aligned} w_j &= z_j - z_{j+1},\ 1 \leq j \leq n-1, \\ y &= \frac{1}{n}\sum_{j=1}^{n} z_j\ , \end{aligned} \tag{9}$$

where e_j is the usual unit vector in $\mathbb{R}^n$ with zeros except for 1 at the j^{th} position and $e_j = \sum_{i=1}^{j} e_i - \frac{j}{n}e$, with $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{n-1})$. The set $e, \tilde{e}_j, 1 \leq j \leq n-1$ is an orthogonal basis for $\mathbb{R}^n$. With this transformation Equation (4) becomes

$$\begin{aligned} \dot{y}(t) &= k(n-1)\big(y(t-r) - y(t)\big) + \frac{1}{n}\sum_{j=1}^{n} g\big(z_j(t)\big), \\ \dot{w}_j(t) &= -k\big((n-1)w_j(t) + w_j(t-r)\big) + \frac{1}{2}G_j(w,y),\ \ j = 1,2,...,n-1, \end{aligned} \tag{10}$$

where $G_j(w,y) = g\big(z_j(t)\big) - g\big(z_{j+1}(t)\big)$.

3. Main Results

In order to prove the theorem on the stability of a synchronization manifold with a small delay, we need to state a lemma that will be useful in the estimate of the delay r for which stability synchronized manifold can be deduced from that of a system without a delay $(r = 0)$. The proof of this lemma can be found in Halanay (1996).

Lemma 1. (Halanay (1996)) If $\dot{f}(t) \leq -\alpha\, f(t) + \beta \sup_{t-r\leq\sigma\leq t} f(\sigma)$ *for* $t \geq t_0$ *and if* $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that* $f(t) \leq Ke^{-\gamma(t-t_0)}$ *for all* $t \geq t_0$ (*Halanay* (*1996*) *pg 378*).

Theorem 2. *Consider equation* (2) *and assume it satisfies the hypotheses* $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$. *Then there is an* r_0 *such that for any* $n \geq 2,$ *and* $K > 0,$ *equation* (2) *has a stable synchronized solution for all* $0 < r < r_0$.

Proof. *For the synchronized manifold to be stable, all transverses to it must asymptotically dampen out. This is equivalent to saying the zero solution of the second equation in* (10) *must be exponentially stable. Linearization of* (10) *about* $\big(y(t), w(t)\big) = \big(y_0(t), 0\big)$ *gives*

$$\begin{aligned} \dot{y}(t) &= \big[J\big(y_0(t)\big) - k(n-1)\big]y(t) + k(n-1)y(t-r), \\ \dot{w}(t) &= \big[J\big(y_0(t)\big) - k(n-1)\big]y(t) + k(n-1)y(t-r), \end{aligned} \tag{11}$$

where $J\left(y_0\left(t\right)\right)$ is the Jacobian of $g\left(z\left(t\right)\right)$ at $\left(y_0\left(t\right),0\right)$ and $w\left(t\right):=w_j\left(t\right)\in\mathbb{R}^d$, $j=1,...,n-1.$

Notice that (11) has a natural splitting $X=X^c\oplus X^s$ each of which is invariant under the linearized flow $DT_k\left(t\right);$ that is, $DT_k\left(t\right)X^p=X^p,\forall t\geq 0,\ 0,p\in\{c,s\}$.

Consider the second equation in Equation (11). If the large r is small, it is quite natural to suppose that it can be neglected and we consider the system of Ordinary Differential Equations (ODEs).

$$\dot{w}\left(t\right)=A\left(t\right)w\left(t\right)+Hw\left(t\right), \tag{12}$$

where $A\left(t\right)=J\left(y_0\left(t\right)\right)-k\left(n-1\right)I$ and $H=-k\left(n-1\right)I$ with I a $d\times d$ identity matrix. Let us suppose that the trivial solution of equation (12) is uniformly asymptotically stable. We could expect that for r sufficiently small the trivial solution of

$$\dot{w}\left(t\right)=A\left(t\right)w\left(t\right)+Hw\left(t-r\right) \tag{13}$$

would be likewise asymptotically stable. This is indeed true for some values of r . This result indicates that the synchronized manifold will also be stable for sufficiently small values of r .

Suppose that the trivial solution $w\left(t\right)=w_j=z_j\left(t\right)-z_{j+1}\left(t\right)=0,\forall\ 1\leq j\leq n-1,$ of equation (12), is uniformly asymptotically stable. If $W\left(t,s\right)$ is the fundamental matrix solution of equation (12), we then have

$$\left\|W\left(t,s\right)\right\|\leq Le^{-\alpha(t-s)},$$

where $L>0,\alpha>0$ are constants. Let $w\left(t\right)$ be any solution of equation (13) then

$$w\left(t\right)=W\left(t,0\right)w\left(0\right)+\int_0^t W\left(t,s\right)H\left[w\left(s-r\right)-w\left(s\right)\right]ds\ . \tag{14}$$

When $t>r$ we can write equation (14) as

$$w\left(t\right)=W\left(t,0\right)w\left(0\right)+\int_0^r W\left(t,s\right)H\left[w\left(s-r\right)-w\left(s\right)\right]ds+\int_r^t W\left(t,s\right)H\left[w\left(s-r\right)-w\left(s\right)\right]ds$$

We know from equation (12) and Gronwall's Lemma that for $0\leq s\leq r$ the estimate :

$$\left|w\left(s\right)\right|\leq\exp\left[r\left(K_0+K_1\right)\right]\left|\varphi\right|,$$

where φ is the initial function of the solution w given on $\left[-r,0\right]$, that

$$K_0=\sup_{0\leq t\leq r}\left\|A\left(t\right)\right\|,\ K_1=\sup_{0\leq t\leq r}\left\|H\right\|.$$

It follows that for $0\leq s\leq r$ we have, in any case,

$$\left|w\left(s-r\right)-w\left(s\right)\right|\leq 2\exp\left[r\left(K_0+K_1\right)\right]\left|\varphi\right|.$$

For $s\geq r$ we may write

$$w\left(s-r\right)-w\left(s\right)=\int_s^{s-r}\dot{w}\left(\sigma\right)d\sigma=\int_s^{s-r}\left[A\left(\sigma\right)w\left(\sigma\right)+Hw\left(\sigma-r\right)\right]d\sigma\ .$$

Hence

$$\left|w(s-r)-w(s)\right| \le r\left(K_0+K_1\right) \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right|,$$

and thus obtain

$$\begin{aligned}\left|w(t)\right| \ \le\ & Le^{-\alpha t}\left|\varphi\right|+\int_0^r Le^{-\alpha(t-s)}K_1 2\exp\left[r\left(K_0+K_1\right)\right]\left|\varphi\right| ds \\ & +\int_r^t Le^{-\alpha(t-s)}K_1 r\left(K_0+K_1\right) \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right| ds \\ & = Le^{-\alpha t}\left|\varphi\right|+2LK_1\exp\left[r\left(K_0+K_1\right)\right]e^{-\alpha t}\frac{1}{\alpha}\left(e^{\alpha r}-1\right)\left|\varphi\right| \\ & +LK_1 r\left(K_0+K_1\right)e^{-\alpha t}\int_r^t e^{\alpha s} \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right| ds \\ & = Le^{-\alpha t}\left|\varphi\right|\left(1+\frac{2}{\alpha}K_1\exp\left[r\left(K_0+K_1\right)\right]\left(e^{\alpha r}-1\right)\right)\end{aligned} \tag{15}$$

Let

$$L_0 = L\left|\varphi\right|\left(1+\frac{2}{\alpha}\left(e^{\alpha r}-1\right)\right)K_1\exp\left[r\left(K_0+K_1\right)\right],$$

and

$$M = LK_1 r\left(K_0+K_1\right).$$

Then equation (15) gives the estimate

$$\left|w(t)\right| \le L_0 e^{\alpha t}+M\int_r^t e^{-\alpha(t-s)} \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right| ds.$$

Let

$$v(t)=e^{-\alpha t}\left[L_0+M\int_r^t e^{\alpha s} \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right| ds\right].$$

Then, we have

$$\begin{aligned}\dot{v}(t) &= -\alpha e^{-\alpha t}\left[L_0+M\int_r^t e^{\alpha s} \sup_{s-r\le\sigma\le s}\left|w(\sigma)\right| ds\right]+e^{\alpha t}Me^{\alpha t} \sup_{t-r\le\sigma\le t}\left|w(\sigma)\right| \\ &= -\alpha v(t)+M \sup_{t-r\le\sigma\le t}\left|w(\sigma)\right|\end{aligned} \tag{16}$$

However, $\left|w(t)\right| \le v(t)$. Hence by equation (16)

$$\sup_{t-r\le\sigma\le t}\left|w(\sigma)\right| \le \sup_{t-r\le\sigma\le t} v(\sigma).$$

Thus

$$.v(t) \le -\alpha v(t)+M \sup_{s-r\le\sigma\le s} v(\sigma)$$

If $M<\alpha,$ it follows by lemma 1 that there exist constants N and γ such that

$$v(t) \le Ne^{-\gamma(t-t_0)}$$

Consequently, the trivial solution of equation (13) is exponentially stable provided that $M < \alpha$. This condition leads to

$$r < \frac{\alpha}{LK_1\left(K_0 + K_1\right)}.$$

Taking

$$r_0 = \frac{\alpha}{KK_1\left(K_0 + K_1\right)},$$

completes the proof of our proposition.

4. Conclusion

We have shown that if a system of equations without delay has a stable synchronization manifold, the introduction of small delays does not affect the stability of the manifold.

5. References

AFRAIMOVICH, V.S. BYKOV, V. and SHILNKOV, L. 1983. On Structurally Unstable Attracting Limit sets of Lorenz attractor Type, *Tran. Of Moscow Math. Soc*. **44:** 153-216.

AFRAIMOVICH, V.S, VERCHEV, N.N. and RIABONVICH, M.I. 1986. Stochastic Synchronization of oscillations in dissipative systems. *Radio Phy. Quantum Electron.* **29:** 37-49

BOSE, F.G. 1989. Stability Conditions for the General Limear Difference – Differential Equation with Constant Coefficients and One Constant Delay. *J. Math. Anal. Appl*. **140:** 136-176.

CHOW, S.N., LIN, W. 1997. Synchronization, stability and normal hyperbolicity. *Resenhas IME-USP*. **3:** 139-158.

HALANAY A. 1996. *Differential equation, stability of oscillators Time-l*ags. Academic press.

HALE, J. 1997. Diffusive coupling, dissipation and synchronization. *J. Dyn. Differ. Equ*. **9:** 1-52.

HALE, J. 1996. Attracting, Manifolds for evolutionary equations *CDSNS* 96-257.

ROSSONI, E., CHEN, Y., DING, M., FENG, J. 2005. Stability of synchronous oscillations in systems of HH neurons with delayed diffusive and pulsed Coupling, *Phys. Rev.* **E 71:** 061904.

GRAHAM, A. 1981. *Kronecker Products and Matrix Calculus with Applications*. Ellis Horwood Limited.

WASIKE, A.A.M., OGANA, W. 2002. Periodic solutions of a system of delay differential equations for a small delay *Science and Technology*, **7:** 295-302.

WASIKE, A.A.M. 2002. Periodic solutions of systems of delay differential equations *Indian Journal of Mathematics,* **44** No. **1:** 95-117.

WASIKE, A.A.M. 2003. Synchronization and oscillator death in diffusively coupled lattice oscillators *International Journal of Mathematical Science,* **2(1):** 67-82.

WASIKE, A.A.M., ROTICH, P.T. 2007. Synchronization, Persistence in diffusively coupled lattice oscillators *SQU Journal for Science,* **12(1):** 41-52.

Received 23 October 2007
Accepted 8 April 2009