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Incidence Matrices of X-Labeled Graphs and an Application

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مصفوفات الوقوع للبيانات المحمولة بعناصر المجموعة - X مع تطبيق

وضاح س. جاسم

خلاصة: في بحثنا هذا قمنا ببعض التحسينات على تعريف مصفوفات الوقوع للبيانات الموجهة من أجل أن تكون أكثر ملاءمة للبيانات المحمولة بعناصر المجموعة – X . بهذه التحسينات استطعنا وصف برنامج لخوارزمية – نكواص.

ABSTRACT: In this work, we have made some modifications to the definition of the incidence matrices of a directed graph, to make the incidence matrices more suitable for X – Labeled graphs. The new incidence matrices are called the incidence matrices of X – Labeled graphs, and we have used the new definition to give a computer program for Nickolas` Algorithm .

KEYWORDS: Incidence matrix, Directed graph, X -labeled graph, Spanning tree, Cyclomatic number.

1. Introduction

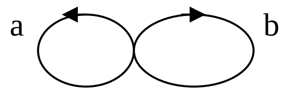
In Abdu (1999), Nickolas described an algorithm to change two core graphs of one type of branch points to core graphs with two or more types of branch points. Since incidence matrices of directed graphs do not deal with labeled graphs, from this point we worked to make the incidence matrices to be more suitable for X – Labeled graphs. This work is divided into three sections. In section 1, we give basic concepts about free groups and graphs. In section 2, we give the definition of the incidence matrices of X – Labeled graphs, and some definitions and results on incidence matrices of X – Labeled graphs. In section 3, we apply the concept of incidence matrices of X – Labeled graphs to give a computer program for Nickolas` Algorithm.

1.1 Basic concepts:

Let F be a group and X be a subset of F. Then we say that F is a *free group* on X, if for any group G and any mapping $f: X \to G$, there is a unique homomorphism $\psi: F \to G$, such that $\psi(x) = f(x)$ for all $x \in X$. Two free groups F and K are called isomorphic if and only if F and K have the same rank.

If a graph Γ is a collection of two sets, $V(\Gamma)$ ($V(\Gamma)$ is not empty set) and $E(\Gamma)$, called the set of vertices and edges respectively of the graph Γ, together with two functions $i : E(\Gamma) \to V(\Gamma), t : E(\Gamma) \to V(\Gamma)$, we say that the edge e joins the vertex $i(e)$ to the vertex $t(e)$. The vertex $i(e)$ is called the initial vertex of e and $t(e)$ is called the terminal vertex of e. Moreover for each e in $E(\Gamma)$ there is an element $\bar{e} \neq e$ in $E(\Gamma)$, which is called the inverse of e, such that $i(\bar{e}) = t(e), t(\bar{e}) = i(e)$ and $\bar{\bar{e}} = e$. A subgraph Ω of a graph Γ is a graph with $V(\Omega) \subseteq V(\Gamma)$ and $E(\Omega) \subseteq E(\Gamma)$ such that, if $e \in E(\Omega)$, then $i_\Omega(e)$, $t_\Omega(e)$ and $\bar{e}$ have the same meaning in Γ as they do in Ω. If $\Omega \neq \Gamma$, then we call Ω a proper subgraph. A component of a connected graph Γ is a maximal connected subgraph of Γ. The number of edges incident with the vertex v is called the degree of the vertex v and denoted by $d(v)$. The vertex v is called a branch point if $d(v) \geq 3$.

Now let F be a group and X be a subset of F. Then the graph $\Gamma(F, X)$ is called the Cayley graph of the group F with respect to X, if $\Gamma(F, X)$ has vertex set F and set of edges $F \times X = \{(w, x) : w \in F, x \in X\}$, such that the initial vertex of the edge (w, x) is $i(w, x) = w$, the terminal vertex of the edge (w, x) is $t(w, x) = wx$ and x is the labeled of the edge (w, x). The inverse edge of (w, x) is (wx, x^{-1}). The quotient graph or Cayley coset graph $\Gamma(F, X)/H$ for a subgroup H of F has set of vertices $\{Hw : w \in F, H \leq F\}$ and set of edges $\{(Hw, x) : w \in F, x \in X\}$ such that an edge $(Hw, x) \in \Gamma(F, X)/H$ takes the vertex Hw to Hwx. It is also denoted by $\Gamma(H)$. The Core of a coset graph $\Gamma(H)$ is the smallest subgraph containing all cycles. It is denoted by $\Gamma^*(H)$. For example, if F is a free group on generators a and b, then

$\Gamma^*(F)$:


The number of cycles in $\Gamma^*(H)$ is called the cyclomatic number and the cyclomatic number of $\Gamma^*(H)$ is the minimal number of edges that we can delete to make a tree. The rank of the finitely generated subgroup H of a free group on X is the cyclomatic number of $\Gamma^*(H)$ and denoted by $r(H)$.

Definition 1.1 A consistent graph is a directed X – Labeled graph Γ on $X = \{a, b\}$, such that no reduced path in Γ with labeled aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$ ever occurs in consistent graphs. Therefore $\Gamma(F, X)$, $\Gamma(H)$ and $\Gamma^*(H)$ are consistent graphs.

Now if $\Gamma^*(H)$ has vertices of degree 2, 3, 4, then as in [5] we can reduce the degree of the vertices into vertices of degree 2 and 3 only, by isomorphic embedding of F into a free group K on $\{u, v\}$, via the map $\theta : F \to K$ with $\theta(a) = uv^{-1}, \theta(b) = v^2$ and taking the graph into a new set of labels $\{u, v\}$.

Proposition 1.2: If $\Gamma^*(H)$ has vertices of degree 2, 3 only, then $r(H) = \dfrac{\# Br(\Gamma^*(H))}{2} + 1$, where $\# Br(\Gamma^*(H))$ is the number of branch points in $\Gamma^*(H)$.

Proof: $\sum d(v) = 2\# E(\Gamma^*(H))$.

$\sum (d(v) - 2) = 2(\# E(\Gamma^*(H)) - \# V(\Gamma^*(H)))$, and the number of edges in the spanning tree of $\Gamma^*(H)$ = $\# V(\Gamma^*(H)) - 1$. Therefore we have the following : $\sum (d(v) - 2)$ = (the number of edges in $\Gamma^*(H)$ - the number of edges in the spanning tree of $\Gamma^*(H)$ - 1). Since $r(H) =$ the number of edges in $\Gamma^*(H)$ - the number of edges in the spanning tree of $\Gamma^*(H)$, so $\sum (d(v) - 2)$ = $2(r(H) - 1) \quad \cdots (1)$, and also

$$d(v) - 2 = \begin{cases} 1 & if \quad v \quad is\, a \quad branch \quad point \\ 0 & otherwise \end{cases}.$$

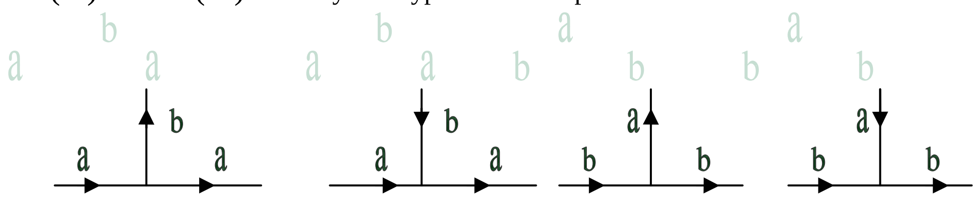
Thus

$$\sum (d(v) - 2) = \# Br(\Gamma^*(H)).\ \text{Therefore}\ r(H) = \frac{\# Br(\Gamma^*(H))}{2} + 1.$$

Definition 1.3: For any two branch points u and v in $\Gamma^*(H)$, we say that u and v are neighboring branch points if they are connected by a (reduced) path which does not contain any branch point.

Definition 1.4: The product of core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ is the graph $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ with set of vertices $V(\Gamma^*(H)) \times V(\Gamma^*(K)) = \{(u,v) : u \in V(\Gamma^*(H)), v \in V(\Gamma^*(K))\}$ and edges $\{((u,v), y) : (u,y) \in E(\Gamma^*(H)), (v,y) \in E(\Gamma^*(K)), y \in X\}$.

If Γ^*(H), $\Gamma^*(K)$ and $\Gamma^*(H \cap K)$ are the core graphs of the finitely generated subgroups H, k and $H \cap K$ respectively of a free group F on $X = \{a, b\}$ and $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ is the product of core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ defined above, then $\Gamma^*(H \cap K)$ may identified with a core of a connected component of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ and $\Gamma^*(H)$ has only four types of branch points as follows:



b – sources $\quad$ b – sinks $\quad$ a – sources $\quad$ a - sinks

2. Incidence matrices of X –Labeled graphs

In this section we will give the definition of incidence matrices of X – Labeled graphs and some definitions and results related to it. As we know there are two types of matrices to describe graphs, which are called adjacency (or vertex incidence) matrices and incidence matrices. Recall that the incidence matrices of directed graphs Γ are without loops and with n vertices and m edges (i.e. it is $n \times m$ matrices $[x_{ij}]$, where $1 \le i \le n, 1 \le j \le m$) such that :

$$x_{ij} = \begin{cases} 1 \quad if \quad \quad v_i = i(e_j) \\ 0 \quad if \quad v_i \; is \quad not\, incedence \quad with \quad e_j \\ -1 \quad if \quad \quad v_i = \tau(e_j) \end{cases}$$

All edges e in $X-$ Labeled graphs $(\Gamma(H), \Gamma^*(H), \Gamma(F,X), \Gamma^*(H) \tilde{\times} \Gamma^*(K), \ldots)$ are labeled $x \in X \cup X^{-1}$ and the incidence matrices of directed graphs do not deal with the labels of edges, so we will put more conditions on the incidence matrices of directed graphs as below.

Definition 2.1: Let Γ be any X – Labeled graph without loops (where $X = \{a,b\}$) , then the incidence matrix of the X – Labeled graph Γ is an $n \times m$ incidence matrix $[x_{ij}]$, where $1 \le i \le n, 1 \le j \le m$) with x_{ij} entries such that

$$x_{ij} = \begin{cases} x \;\; if \; v_i = \quad i(e_j) \quad and \quad e_j \quad \quad lables \quad x \in X \\ \quad 0 \; if \; v_i \quad is \quad not\; incident \quad with \quad e_j \\ x^{-1} \; if \; v_i = \quad \tau(e_j) \quad and \quad e_j \quad \quad labeles \quad x \in X \end{cases}$$

N.B. Incidence matrices of X – Labeled graphs Γ will be denoted by $M_X(\Gamma)$.

Now if $X = \{a,b\}$ and the X –Labeled graph Γ has loops with labeling a or b, then choose a mid point on all edges labeled a or b to make all of them two edges labeled aa or bb respectively. Therefore in the rest of this work we will assume that all X – Labeled graphs Γ are without loops.

Definition 2.2: Let $M_X(\Gamma)$ be an incidence matrices of X – Labeled graphs Γ. If $M_X(\Gamma)$ does not contain any row r_i with non zero entries x_{ij} and x_{ik} in $X \cup X^{-1}$ such that $x_{ij} = x_{ik}$, then $M_X(\Gamma)$ is called a consistent incidence matrix of X – Labeled graphs Γ.

Now let $M_X(\Gamma)$ be an $n \times m$ incidence matrix $[x_{ij}]$ of X – Labeled graphs Γ and let r_i and c_j be a row and a column in $M_X(\Gamma)$ respectively. If x_{ij} is a non – zero entry in the row r_i, then r_i is called an incidence row with the column c_j at the non – zero entry $x_{ij} \in X \cup X^{-1}$ and if $x_{ij} \in X$, then the row r_i is called the starting row (denoted by $s(c_j)$) of the column c_j and the row r_i is called the ending row (denoted by $e(c_j)$) of the column c_j if $x_{ij} \in X^{-1}$. If the rows r_i and r_k are incident with column c_j at the non – zero entries x_{ij} and x_{kj} respectively, then we say that the rows r_i and r_k are adjacent. If c_j and c_h are two distinct columns in $M_X(\Gamma)$ such that the row r_i is incident with the columns c_j and c_h at the non – zero entries x_{ij} and x_{ih} respectively (where $x_{ij}, x_h \in X \cup X^{-1}$), then we say that c_j and c_h are adjacent columns. For each column c there is an inverse column denoted by $\bar{c}$ such that $s(\bar{c}) = e(c), e(\bar{c}) = s(c)$ and $\bar{\bar{c}} = c$. The degree of a row r_i of $M_X(\Gamma)$ is the number of the columns incident on r_i and is denoted by $\deg(r_i)$. If the row r_i is incident with at least three distinct columns c_j, c_h and c_k at the non – zero entries, then the row r_i is called a branch row. If the row r_i is incident with only one column c_j at the non zero entry $x_{ij} \in X \cup X^{-1}$ and all other entries of r_i are zero, then the row r_i is called isolated row. A scale in $M_X(\Gamma)$ is a finite sequence of form $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \dots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$, where $k \geq 1, \in = \mp$, $s(c_j^{\in_j}) = r_j$ and $e(c_j^{\in_j}) = r_{j+1} = s(c_{j+1}), 1 \leq j \leq k$. The starting row of a scale $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \dots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$ is the starting row r_1 of the column c_1 and the ending row of the scale S is the ending row r_k of the column c_{k-1} and we say that S is a scale from r_1 to r_k and S is a scale of length k for $1 \leq j \leq k-2$. If $s(S) = e(S)$, then the scale is called closed scale. If the scale S is reduced and closed, then S is called a circuit or a cycle. If $M_X(\Gamma)$ has no cycle, then $M_X(\Gamma)$ is called a forest incidence matrix of X – Labeled graph Γ. Two rows r_i and r_k in $M_X(\Gamma)$ are called connected if there is a scale S in $M_X(\Gamma)$ containing r_i and r_k. Moreover, $M_X(\Gamma)$ is called connected if any two rows r_i and r_k in $M_X(\Gamma)$ are connected by a scale S. If $M_X(\Gamma)$ is connected and forest, then $M_X(\Gamma)$ is called a tree incidence matrix of X – Labeled graph Γ. Let Ω be a subgraph of Γ, then $M_X(\Omega)$ is called a subincidence matrix of $M_X(\Gamma)$, if the set of rows and columns of $M_X(\Omega)$ are subsets of $M_X(\Gamma)$ and if c is a column in $M_X(\Delta)$. Then $s(c), e(c)$ and $\bar{c}$ have the same meaning in $M_X(\Gamma)$ as they do in $M_X(\Omega)$. If $M_X(\Omega) \neq M_X(\Gamma)$, then $M_X(\Omega)$ is called a proper subincidence matrix of $M_X(\Gamma)$. A component of $M_X(\Gamma)$ is a maximal connected subincidence matrix of $M_X(\Gamma)$. If $M_X(\Omega)$ is a subincidence matrix of $M_X(\Gamma)$, and every two rows r_i and r_k in $M_X(\Gamma)$ are joined by at least one scale S in $M_X(\Omega)$, then $M_X(\Omega)$ is called a spanning incidence matrix of $M_X(\Gamma)$, and $M_X(\Omega)$ is called a spanning tree of $M_X(\Gamma)$ if $M_X(\Omega)$ is a spanning and tree incidence matrix. The inverse of $M_X(\Gamma)$ is an incidence matrix of X^{-1} - labeled graph Γ.

Now by direct calculations and the definition above, we can prove the following results.

Lemma 2.3: If $M_X(\Gamma)$ is a tree incidence matrix of X – Labeled graph Γ with n rows, then $M_X(\Gamma)$ has $n - 1$ columns.

Lemma 2.4: If $M_X(\Gamma)$ is an incidence matrix of X – Labeled graph Γ with n rows and m columns, then $\sum_{i=1}^{n} \deg(r_i) = 2m$, where $1 \leq i \leq n$.

Corollary 2.5: If $M_X(\Gamma)$ is a finite incidence matrix of X – Labeled graph Γ, then $M_X(\Gamma)$ has even number of rows of odd degree.

Lemma 2.6: The cyclomatic number $C(M_X(\Gamma))$ of a finite incidence matrix of X – Labeled graph Γ is equal to $\#c - \#r + k$, where $\#c$, $\#r$ and k are the number of columns, the number of rows and the number of components of $M_X(\Gamma)$ respectively.

Corollary 2.7: If $M_X(\Gamma)$ is a connected incidence matrix of X – Labeled graph Γ with r rows and c columns, then $C(M_X(\Gamma)) = \#c - \#r + 1$.

3. An application of incidence matrices of X – Labeled graphs

In this section we will represent the core graphs $\Gamma^*(H)$ of finitely generated subgroups H of a free group F generated by $X=\{a,b\}$ in the form of incidence matrices of core graphs $M_X(\Gamma^*(H))$ and then we will describe Nickolas` Algorithm (Nickolas, 1985) in the form of incidence matrices of X – Labeled graphs, but before that we give some results and definitions that will be used in the rest of the paper.

Since the core graphs $\Gamma^*(H)$ of finitely generated subgroups H of a free group F generated by $X=\{a,b\}$ are X – Labeled graphs, they may be represented by incidence matrices of X – Labeled graphs Γ which are denoted by $M_X(\Gamma^*(H))$.

Lemma 3.1: Let $M_X(\Gamma^*(H))$ be an incidence matrix of a core graph $\Gamma^*(H)$ of a finitely generated subgroup H of a free group F generated by $X=\{a,b\}$ with n rows of degree two and three only and m columns, then $C(M_X(\Gamma^*(H)))=\dfrac{\#Br(M_X(\Gamma^*(H)))}{2}+1$, where $\#Br(M_X(\Gamma^*(H)))$ is the number of branch rows of $M_X(\Gamma^*(H))$.

Proof: Since $\sum_{i=1}^{n}\deg(r_i)=2m$, so $\sum(\deg(r_i)-2)=2(m-n)$.

But $\sum(\deg(r_i)-2)=\#Br(M_X(\Gamma^*(H)))$ and $C(M_X(\Gamma^*(H)))-1=m-n$.

Therefore $C(M_X(\Gamma^*(H)))=\dfrac{\#Br(M_X(\Gamma^*(H)))}{2}+1$.

Corollary 3.2: Let $M_X(\Gamma^*(H))$ be an incidence matrix of a core graph $\Gamma^*(H)$ of a finitely generated subgroup *H* of a free group *F* generated by $X=\{a,b\}$ with n rows of degree two and three only, then $r(H)=\dfrac{\#Br(M_X(\Gamma^*(H)))}{2}+1$.

Definition 3.3: Let $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ be the incidence matrices of core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ respectively, of a finitely generated subgroup *H* of a free group *F* generated by $X=\{a,b\}$. The product of $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ is the incidence matrix $M_X(\Gamma^*(H))\,\tilde{\times}\,M_X(\Gamma^*(K))$ of *X* – Labeled graph of the product of $\Gamma^*(H)$ and $\Gamma^*(K)$ with set of rows $\{(v,u):v$ and u are rows in $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ respectively $\}$ and set of columns $\{(c_i,c_j):$ c_i and c_j are columns in $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$,respectively$\}$, and they have the same non – zero entries $x\in X\cup X^{-1}$.

Lemma 3.4: Let $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ be the incidence matrices of core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ respectively, of a finitely generated subgroup *H* of a free group *F* generated by $X=\{a,b\}$. Then the incidence matrices of the product $\Gamma^*(H)\tilde{\times}\Gamma^*(K)$ (denoted by $M_X(\Gamma^*(H)\tilde{\times}\Gamma^*(K))$) is the same as the product $M_X(\Gamma^*(H))\tilde{\times}M_X(\Gamma^*(K))$.

Proof: By the definition of the product of core graphs and definition 3.3 the result follows.

Lemma 3.5: Let $M_X(\Gamma^*(H))$ be defined as above and have 2*n* branch rows. If all branch rows of $M_X(\Gamma^*(H))$ are of one type *b* – sources, say , then there are at least *n* rows with only two non – zero entries $x_{ij} = b^{-1}, x_{ik} = a$ and all other entries are zero.

Proof: Since all possibilities of scales joining two neighboring branch rows are $S_1, S_2, S_3, S_4, S_5, S_6$ such that $S_1 = r_{i_1}c_{j_1}r_{i_2}c_{j_2}\cdots r_{i_k}c_{j_k}r_{i_{k+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non – zero entry $x_{i_1j_1} = b$ and $r_{i_{k+1}}$ is the ending row of the columns c_{j_k} at the non – zero entry $x_{i_{k+1}j_{k+1}} = a^{-1}$, $S_2 = r_{i_1}c_{j_1}r_{i_2}c_{j_2}\cdots r_{i_m}c_{j_m}r_{i_{m+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non – zero entry $x_{i_1j_1} = b$ and $r_{i_{m+1}}$ is the ending row of the column c_{j_m} at the non–zero entry $x_{i_{m+1}j_m} = a$, $S_3 = r_{i_1}c_{j_1}r_{i_2}c_{j_2}\cdots r_{i_t}c_{j_t}r_{i_{t+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non – zero entry $x_{i_1j_1} = b$ and $r_{i_{t+1}}$ is the ending row of the columns c_{j_k} at the non–zero entry $x_{i_{t+1}j_t} = b$; $S_4 = r_{i_1}c_{j_1}r_{i_2}\cdots r_{i_g}c_{j_g}r_{i_{g+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non – zero entry $x_{i_1j_1} = a$ and $r_{i_{g+1}}$ is the ending row of the columns c_{j_g} at the non–zero entry $x_{i_{g+1}j_g} = a^{-1}$, $S_5 = r_{i_1}c_{j_1}r_{i_2}\cdots r_{i_h}c_{j_h}r_{i_{h+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non–zero entry $x_{i_1j_1} = a$ and $r_{i_{t+1}}$ is the ending row of the columns c_{j_k} at the non-zero entry $x_{i_{t+1}j_t} = a$, and $S_6 = r_{i_1}c_{j_1}r_{i_2}\cdots r_{i_d}c_{j_d}r_{i_{d+1}}$; where r_{i_1} is the starting row of the column c_{j_1} at the non – zero entry $x_{i_1j_1} = a^{-1}$ and $r_{i_{t+1}}$ is the ending row of the columns c_{j_k} at the non–zero entry $x_{i_{t+1}j_t} = a^{-1}$.

Since $M_X(\Gamma^*(H))$ is a consistent incidence matrix of *X* – Labeled graph Γ, so the scales S_1, S_2, S_3, S_4, S_5 and S_6 must contain rows r_{i_f} with a non – zero entries $x_{i_fj_v} = b^{-1}$, and $x_{i_fj_u} = a$ or $x_{i_fj_u} = a^{-1}$ only and all other entries are zero. Also $M_X(\Gamma^*(H))$ has 2*n* branch rows and all of them are of one type b – source row, so suppose that $M_X(\Gamma^*(H))$ has s_1 scales of type S_1, s_2 scales of type S_2, s_3 scales of type S_3, s_4 scales of type S_4, s_5 scales of type S_5 and s_6 scales of type S_6. Therefore we have $s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3n \cdots(1)$. Since each branch row r_i of $M_X(\Gamma^*(H))$ has exactly one non – zero entry $x_{ij} = a$, and the scale of type S_2 ending with the row r_{i_m} at the non - zero entries $x_{i_mj_m} = a$, scales of types S_4 starting with the row r_{i_1} at the non – zero entry $x_{i_1j_1} = a$ and scales of types S_5 starting and ending with the rows r_{i_1} and r_{i_h} at the non – zero entries $x_{i_1j_1} = a$ and $x_{i_1j_h} = a$ respectively. Therefore $M_X(\Gamma^*(H))$ has $s_2 + s_4 + 2s_5 = 2n\cdots(2)$. Hence from (2) we have $s_2 + s_4 + s_5 < 2n\cdots(3)$. From (3) and (1) we have $s_1 + s_3 + s_6 > n$, since each of the scales S_1, S_3 and S_6 has at least one row with non – zero entries $x_{ij} = b^{-1}$ and $x_{iz} = a$, and all other entries are zero.

Nickolas [5] gave an algorithm to change core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ of two finitely generated subgroups H and K of a free group F generated by $X=\{a,b\}$, which have only one type of branch points, b – sources,say, into new core graphs with two or more types of branch points. Therefore we will represent Nickolas` algorithm in the form of incidence matrices of X –Labeled graphs in order to give a computer program to change the type of branch points of core graphs with only one type of branch points into two or more types of branch points.

Now let $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ be incidence matrices of X –Labeled core graphs $\Gamma^*(H)$ and $\Gamma^*(K)$ respectively, such that $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ have branch rows of one type, b – sources, say, then we will use the representation of Nickolas` algorithm to change $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ into two incidence matrices of X –Labeled core graphs $M_X(\Gamma_k^*(H))$ and $M_X(\Gamma_k^*(K))$ with two types or more of branch rows, after k – times . The steps are given below:

0) Delete all zero columns and zero rows if they appear;

0^*) If the branch rows are not of one type, then stop. Otherwise, change the non – zero entries to make all branch rows of type b – sources by reversing the labeling of the columns and then proceed to step 1;

1) If r_i is the ending and the starting row of the columns c_j and c_k at $x_{ij}=b^{-1}$ and $x_{ik}=b$ respectively and all other entries of r_i are zero, and r_t is the ending row of the column c_k, then add r_t and r_i to a new row $r_{t'}$, and if there is no such a row r_i, then proceed to step II;

II) If r_i, r_g and r_t are rows such that r_i is the ending and the starting row of the columns c_j and c_k at $x_{ij}=b^{-1}$ and $x_{ik}=a$ respectively, and r_g is the ending row of the columns c_k and c_h at $x_{gk}=a^{-1}$ and $x_{gh}=b^{-1}$ respectively and all other entries of r_i and r_g are zero, and also r_t is the starting row of the column c_h at $x_{th}=b$, then add r_t, r_i and r_g to have a new row $r_{t'}$, and if there is no such rows, then proceed to step III;

III) If r_i is the ending and the starting row of the columns c_j and c_k at $x_{ij}=b^{-1}$ and $x_{ik}=a$ respectively and all other entries of r_i are zero, and r_t is the ending row of the column c_k at $x_{tk}=a^{-1}$, then add r_t and r_i to have a new row $r_{t'}$, and if there is no such a row then return to step 0 above.

Note: In the program of the above Algorithm we will consider a^{-1} and b^{-1} as $-a$ and $-b$ respectively and then will represent a and b by 1 and 2 respectively. The representation of Nickolas` Algorithm in the form of incidence matrices of $X$ –Labeled graphs is well defined, which means the representation of Nickolas` Algorithm satisfying the following conditions:

1) At each step of the representation of Nickolas` Algorithm, we get consistent incidence matrices of X –Labeled graphs because step I is applied to non – branch rows r_i and r_t to give a non – branch row $r_{t'}$ with non – zero entries $x_{t'j}=b^{-1}$ and $x_{t'h}=a$ or $x_{t'h}=a^{-1}$, so step I always gives consistent incidence matrices of X –Labeled graphs . Also step II is applied on non – branch rows of $M_X(\Gamma^*(H))$ which do not contain non – branch rows r_i with non – zero entries $x_{ij}=b^{-1}$ and $x_{ih}=b$ so step II always gives, either non – branch

rows $r_{t'}$ with non – zero entries $x_{t'j} = a^{-1}$ and $x_{t'h} = b^{-1}$ or a branch row with non – zero entries $x_{t'h} = b^{-1}$, $x_{t'j} = a$ and $x_{t'l} = a^{-1}$. While step III is applied on non – branch row r_i with non – zero entries $x_{ij} = b^{-1}$ and $x_{ik} = a$, and the row r_t which is either a non – branch row with non – zero entries $x_{tk} = a^{-1}$ and $x_{th} = b$ or $x_{th} = a$, or r_t is a branch row of type b – sources , so when we apply step III either we have a non - branch row $r_{t'}$ with non zero entries $x_{t'j} = b^{-1}$ and $x_{t'h} = b$ or $x_{t'j} = b^{-1}$ and $x_{t'h} = a$, or $r_{t'}$ is a branch row with non – zero entries $x_{t'j} = b^{-1}$, $x_{t'h} = b$ and $x_{t'l} = a$. Therefore step III always gives consistent incidence matrices of X –Labeled graphs.

2) The representation of Nickolas` Algorithm preserves the number of branch rows in $M_X(\Gamma^*(H))$, because it is applied on non – branch rows, so apply steps I, II and III reduce the non – branch rows only and then the number of branch rows in $M_X(\Gamma^*(H))$ will be still the same as before and the type of branch rows may change only.

3)The representation of Nickolas` Algorithm preserves the number of branch rows in $M_X(\Gamma^*(H)) \tilde{\times} M_X(\Gamma^*(K))$, because each row in $M_X(\Gamma^*(H)) \tilde{\times} M_X(\Gamma^*(K))$ comes from the product of two rows r_i and r_t in $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ respectively and the removed rows are non – branch rows which contain a non zero entries $x_{ij} = b^{-1}$, and $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$ have no branch row with non –zero entry $x_{ij} = b^{-1}$. Therefore whatever happens to $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$, then will happen to the maximum connected submatrix of $M_X(\Gamma^*(H)) \tilde{\times} M_X(\Gamma^*(K))$.

4) Each time we return to step 0^* of the representation of Nickolas` Algorithm, we must apply one of the steps 1, 2 or 3 , because if step 1 and 2 do not apply , the step 3 must apply by Lemma 3.5.

5) In each time we return to step 0^* we have fewer rows ; this comes from reducing at least two rows to a single row.

6) The algorithm must stop after a finite time, because each time we reduce the total number of rows by at least one row, and then we have at least two different types of branch rows of $M_X(\Gamma^*(H))$ and $M_X(\Gamma^*(K))$.

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