

# Local Projection Stabilization Method for Incompressible Flow Problems

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**ABSTRACT:** This paper analyzes Local Projection Stabilization (*LPS*) methods for the solution of Stokes problem using equal order finite elements. Their convergence, stability and accuracy properties are investigated. The resulting stabilized method is shown to lead to optimal rates of convergence for both velocity and pressure approximations. Two classes of *LPS* methods are distinguished: one-level and two-level methods. Numerical examples using bilinear interpolations are presented to validate the analysis and assess the accuracy of both approaches.

**KEYWORDS:** Finite elements, Local projection stabilization, Rates of convergence, Stability.

استقرار طريقة الإسقاط المحلي لحل مسائل السوائل اللزجة الغير قابلة للانضغاط

عفيفة السافي و كمال نافع

**ملخص:** نحلل في هذا البحث طرق الإسقاط الموضوعي لحل مسألة ستوكس باستخدام درجة متساوية لعناصر محدودة. نتحرى تقارب واستقرار تلك الطرق وخصائص دقتها. نبرهن ان طريقة الإسقاط الموضوعي مستقرة وتؤدي إلى أفضل نسب للتقارب سواء بالنسبة للسرعة أو للضغط. نميز بين صنفين من الطرق: إسقاطات ذات مستوى واحد واثنين. نصف أمثلة عددية باستخدام استقرارات شبه خطية للتحقق من صحة التحليل وتقييم مدى دقة كلا النهجين.

## 1. Introduction

Numerical approximations of incompressible flows require a compatibility condition between the discrete velocity and pressure spaces (Girault and Raviart, 1986; Brezzi and Fortin (1991)). This condition prevents, in particular, the use of equal order interpolation spaces for the two variables, which is the most attractive choice from a computational point of view. The two-level local projection technique has been introduced by Becker and Braack (2001) and Nafa (2008) to circumvent the inf-sup condition and to allow the use of simple equal order interpolations such as  $P^1 - P^1$  and  $Q^1 - Q^1$  velocity-pressure approximations.

This method consists in introducing the  $L^2$ -projection of the pressure gradient as a new unknown of the problem. Hence, a third equation to enforce the projection property is added to the original discrete equations,

and a weighted difference of the pressure gradient and its local projection is introduced into the continuity equation.

In this paper we analyze the *LPS* and distinguish two classes of *LPS* methods: The one-level enriched method and the two-level method. The stability and error estimate results presented here are more general than those given in Becker and Braack (2001). In fact, the proofs extend the stability and error estimate results of Becker and Braack (2001) and Nafa (2008) to the one-level enriched approximation case. Numerical results are presented to justify the order of convergence and assess the performance accuracy of both *LPS* methods using bilinear finite element interpolation.

## 2. Discrete Stokes problem

Let  $\Omega$  denote an open, bounded, connected subset of  $\mathfrak{R}^2$ , where  $\partial\Omega$  denotes its Lipschitz continuous boundary. Let

$$\mathbf{f} \in \left[ H^{-1}(\Omega) \right]^2, \mathbf{V} = \left[ H_0^1(\Omega) \right]^2, \text{ and } Q = L_0^2(\Omega).$$

We consider the usual Stokes problem with homogeneous Dirichlet boundary conditions. Find  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  satisfying

$$\begin{aligned} \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial\Omega \end{aligned} \quad (1)$$

where  $L_0^2(\Omega)$  is the set of square integrable functions with null average.

The weak formulation of the above problem reads

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) = (\mathbf{f}, \mathbf{v}) \quad (2)$$

for all test functions  $(\mathbf{v}, q) \in \mathbf{V} \times Q$  where  $(\cdot, \cdot)$  denotes the  $L^2$ -inner product on the region  $\Omega$ . The compact form of the weak formulation of (1) is given by

$$B(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q \quad (3)$$

where

$$B(\mathbf{u}, p; \mathbf{v}, q) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u})$$

and

$$F(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v})$$

## 3. Local projection stabilization

Let  $T_h$  be a shape regular partition of the region  $\Omega$  into quadrilateral elements  $K$ , and assume  $\mathbf{V}_h$  and  $Q_h$  are finite dimensional subspaces of  $\mathbf{V}$  and  $Q$  respectively, consisting of continuous functions. We denote by  $h_k$  the local mesh size, the mesh size is then defined by  $h = \max_{K \in T_h} h_K$ . Then, the Galerkin discrete problem reads

$$B(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = F(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \quad (4)$$

Let  $M_h$  be a coarser mesh partition of the domain into quadrilateral elements  $M$  which are defined as the union of one or more neighboring elements  $K$  of the partition  $T_h$ . We assume that the partition of  $\Omega$  into macro-elements  $M \in M_h$  is non-overlapping, shape regular and such that

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$$h_M \approx h_K, \quad \forall K \subset M, \quad \forall M \in M_h$$

where  $h_K$  and  $h_M$  denote the maximum diameter of  $K$  and  $M$ , respectively.

Let  $Y_h \subset H^1(\Omega)$  be a scalar finite element space of continuous, piecewise polynomial functions over  $T_h$ . Since we are interested in the case of equal order interpolation, we define the approximation spaces as follows

$$\mathbf{V}_h = Y_h^2 \cap \mathbf{V}, \quad Q_h = Y_h \cap Q.$$

Let  $D_h(M)$  be a discontinuous finite dimensional space defined on the macro-element  $M \in M_h$ . We introduce the associated discontinuous global space

$$\mathbf{D}_h = \bigoplus_{M \in M_h} \mathbf{D}_h(M).$$

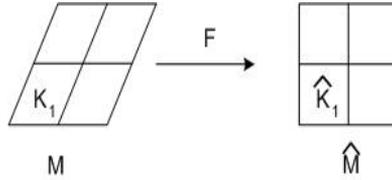


Figure 1. Reference macro-element (right) and macro-element (left).

Then, we define the equal order approximation finite element spaces of velocity and pressure respectively by:

$$\begin{aligned} \mathbf{V}_h &= \left\{ \mathbf{v} \in \left( H_0^1(\Omega) \right)^2 / \mathbf{v}|_K \in \left( Q_r(K) \right)^2, \quad \forall K \in T_h \right\} \\ Q_h &= \left\{ q \in H^1(\Omega) \cap L_0^2(\Omega) / q|_K \in Q_r(K), \quad \forall K \in T_h \right\} \end{aligned} \quad (5)$$

where, for each integer  $r \geq 0$ ,  $Q_r(K)$  denotes standard quadrilateral finite elements by means of a reference element (as illustrated in Figure 1) as polynomials of degree less or equal to  $r$  with respect to each variable. Further, we define the space of discontinuous functions

$$\mathbf{D}_h = \left\{ \boldsymbol{\lambda}_h \in \left( L^2(\Omega) \right)^2 / \boldsymbol{\lambda}_h|_K \in \left( Q_{r-1}(M) \right)^2, \quad \forall M \in M_h \right\}. \quad (6)$$

We define the local  $L^2$ -projection  $\pi_M : L^2(M) \rightarrow \mathbf{D}_h(M)$  which generates the global projection  $\pi_h : L^2(\Omega) \rightarrow \mathbf{D}_h$  by

$$\pi_h(\omega)|_M = \pi_M(\omega_M), \quad \forall M \in M_h, \quad \forall \omega \in L^2(\Omega).$$

To which, we associate the fluctuation operator  $\kappa_h : L^2(\Omega) \rightarrow L^2(\Omega)$  defined as  $\kappa_h = id - \pi_h$ , where  $id$  stands for the identity operator in  $L^2(\Omega)$ . For simplicity we use the same notations  $id$ ,  $\pi_M$ ,  $\pi_h$  and  $\kappa_h$  for vector valued functions.

Now, the stabilizing term is defined by

$$S_h B(p_h, q_h) = \sum_{M \in M_h} \alpha_M (\kappa_h \nabla p_h, \kappa_h \nabla q_h)_{0,M}. \quad (7)$$

Thus, the LPS discrete problem reads: Find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$  such that

$$B_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = F_h(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times \mathcal{Q}_h \quad (8)$$

where

$$B_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = B(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) + S_h B(p_h, q_h), \quad (9a)$$

and

$$F_h(\mathbf{v}_h, q_h) = F(\mathbf{v}_h, q_h). \quad (9b)$$

The stabilized formulation given in (8) is written component-wise as

$$\mathbf{v}(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \quad (10a)$$

$$-(q_h, \nabla \cdot \mathbf{u}_h) + \sum_{M \in \mathcal{M}_h} \alpha_M (\nabla p_h - \pi_h(\nabla p_h), \nabla q_h)_{0,M} = 0, \quad \forall q_h \in \mathcal{Q}_h \quad (10b)$$

$$- \sum_{M \in \mathcal{M}_h} \alpha_M (\nabla p_h - \pi_h(\nabla p_h), \boldsymbol{\xi}_h)_{0,M} = 0, \quad \forall \boldsymbol{\xi}_h \in \mathbf{D}_h. \quad (10c)$$

In order to prove the stability and convergence of the solution of the stabilized method given in (8) we introduce the following assumptions.

**Assumption A1.** The fluctuation operator satisfies the approximation property

$$\|\kappa_h q\|_{0,M} \leq Ch_M^l |q|_{l,M}, \quad \forall q \in H^l(M), \quad \forall M \in \mathcal{M}_h, \quad 0 \leq l \leq r. \quad (11)$$

**Assumption A2.** There exists an interpolation operator  $i_h : H^1(\Omega) \rightarrow Y_h$  such that

$i_h \mathbf{v} : H_0^1(\Omega) \rightarrow Y_h \cap H_0^1(\Omega)$  with the error estimates

$$\|\mathbf{v} - i_h \mathbf{v}\|_{0,K} + h_K |\mathbf{v} - i_h \mathbf{v}|_{1,K} \leq Ch_K^s \|\mathbf{v}\|_{s,\omega(K)} \quad (12)$$

for all  $\mathbf{v} \in H^s(\omega(K))$ , and all  $K \in \mathcal{T}_h$ ,  $1 \leq s \leq r+1$ , where  $\omega(K)$  denotes a certain neighborhood of  $K$ .

**Assumption A3.** Further, assume that the local inf-sup condition

$$\inf_{q_h \in \mathcal{D}_h(M)} \sup_{\mathbf{v}_h \in \mathcal{V}_h(M)} \frac{(\mathbf{v}_h, q_h)_M}{\|\mathbf{v}_h\|_{0,M} \|q_h\|_{0,M}} \geq \beta_1 \quad (13)$$

holds for all  $M \in \mathcal{M}_h$  with a positive constant  $\beta_1$  independent of the mesh size  $h$ .

The following theorem defines  $j_h$  and  $\mathbf{j}_h$  interpolation operators that are important for deriving error estimates for the LPS method (Matthies *et al.* 2007).

**Theorem 1.** Assume that Assumptions A2 and A3 are satisfied. Then, there are interpolation operators  $j_h : H^1(\Omega) \rightarrow Y_h$  and  $\mathbf{j}_h : \mathbf{V} \rightarrow \mathbf{V}_h$  satisfying the following orthogonality and approximation properties

$$(\omega - j_h \omega, \Phi_h) = 0, \quad \forall \Phi_h \in \mathcal{D}_h, \quad \forall \omega \in H^1(\Omega) \quad (14a)$$

$$\|\omega - j_h \omega\|_{0,M} + h_M |\omega - j_h \omega|_{1,M} \leq Ch_M^s \|\omega\|_{s,\omega(M)} \quad (14b)$$

and

$$(\boldsymbol{\omega} - \mathbf{j}_h \boldsymbol{\omega}, \Phi_h) = 0, \quad \forall \Phi_h \in \mathbf{D}_h, \quad \forall \boldsymbol{\omega} \in \mathbf{V} \quad (15a)$$

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$$\|\boldsymbol{\omega} - \mathbf{j}_h \boldsymbol{\omega}\|_{0,M} + h_M \|\boldsymbol{\omega} - \mathbf{j}_h \boldsymbol{\omega}\|_{1,M} \leq Ch_M^s \|\boldsymbol{\omega}\|_{s,\omega(M)}. \quad (15b)$$

For all  $\omega \in H^s(\omega(M))$ ,  $\boldsymbol{\omega} \in \mathbf{V} \cap (H^s(M))^2$ ,  $M \in M_h$ , and  $1 \leq s \leq r+1$ , where  $\omega(M)$  denotes again a certain neighborhood of  $M$ .

Next, we introduce the following mesh-dependent norm on the product space  $\mathbf{V}_h \times Q_h$

$$\|(\mathbf{v}_h, q_h)\|^2 = \nu \left( \|\mathbf{v}_h\|_{1,\Omega}^2 + \|q_h\|_{0,\Omega}^2 \right) + \sum_{M \in M_h} \alpha_M \|\kappa_h \nabla q_h\|_{0,M}^2. \quad (16)$$

### 3.1 Stability

**Theorem 2.** Assume that Assumptions A2 and A3 hold and the parameters  $\alpha_M$  are such that  $h_M^2 / \alpha_M \leq C$  for all elements  $M \in M_h$ . Then, there is a positive constant independent of  $h$  such that

$$\inf_{(\mathbf{0},0) \neq (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h} \sup_{(\boldsymbol{\omega}_h, r_h) \in \mathbf{V}_h \times Q_h} \frac{B_h(\mathbf{v}_h, q_h; \boldsymbol{\omega}_h, r_h)}{\|(\mathbf{v}_h, q_h)\| \cdot \|(\boldsymbol{\omega}_h, r_h)\|} \geq \beta_2. \quad (17)$$

The proof is found in (Nafa and Wathen, 2009).

### 3.2 Error estimates

First, we introduce the following consistency error (Ganesan *et al.*, 2008).

**Lemma 3.** Assume that the fluctuation operator  $\kappa_h$  satisfies Assumption A1. Let

$(\mathbf{u}, p) \in \mathbf{V} \cap [H^{r+1}(\Omega)]^2 \times Q \cap H^r(\Omega)$  be the solution of Stokes problem and  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$  be the solution of the LPS stabilized problem. Then, the consistency error can be estimated by

$$B(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{v}_h, q_h) \leq C \left( \sum_{M \in M_h} \alpha_M h_M^{2r-2} |p|_{r,M}^2 \right)^{1/2} \|(\mathbf{v}_h, q_h)\| \quad (18)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ . Then, the error estimate of the LPS method is given by the following theorem.

**Theorem 4.** Assume that the solution  $(\mathbf{u}, p)$  of (3) belongs to  $\mathbf{V} \cap [H^{r+1}(\Omega)]^2 \times Q \cap H^r(\Omega)$ ,

$(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$  is the solution of the local projection method (8), and  $\alpha_M$  are such that  $h_M^2 / \alpha_M \leq C$  for all elements  $M \in M_h$ . Then, the following error estimates hold

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^r \left( \|\mathbf{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega} \right) \quad (19)$$

for some positive constant  $C$  independent of  $h$ .

**Proof.** Let  $\tilde{\mathbf{u}}_h = j_h u$  and  $\tilde{p} = i_h p$  be the interpolants of the velocity and pressure respectively. By virtue of Theorem 2, there exist  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$  satisfying

$$\|(\mathbf{v}_h, q_h)\| \leq C. \quad (20)$$

We have

$$\|(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h)\|^2 = \nu \left( \|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega}^2 + \|\tilde{p}_h - p_h\|_{0,\Omega}^2 \right) + S(\tilde{p}_h - p_h, \tilde{p}_h - p_h). \quad (21)$$

Hence,

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega}^2 + \|\tilde{p}_h - p_h\|_{0,\Omega}^2 \leq 1/\nu^{1/2} \|(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h)\|. \quad (22)$$

Thus, by equivalence of norms on finite dimensional spaces in  $\mathfrak{R}^2$ , it yields

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega}^2 + \|\tilde{p}_h - p_h\|_{0,\Omega}^2 \leq \sqrt{2}/\nu^{1/2} \|(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h)\|. \quad (23)$$

In addition, we have

$$\|(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h)\| \leq 1/\beta_2 \frac{B_h(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h; \mathbf{v}_h, q_h)}{\|(\mathbf{v}_h, q_h)\|} \quad (24)$$

i.e.

$$\|(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \tilde{p}_h - p_h)\| \leq 1/\beta_2 \frac{B_h(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - p; \mathbf{v}_h, q_h)}{\|(\mathbf{v}_h, q_h)\|} + 1/\beta_2 \frac{B_h(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{v}_h, q_h)}{\|(\mathbf{v}_h, q_h)\|}. \quad (25)$$

Using the result of the consistency estimate with  $\alpha_M = Ch_M^2$ , we obtain

$$\frac{B_h(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{v}_h, q_h)}{\|(\mathbf{v}_h, q_h)\|} \leq Ch^r \|p\|_{r,\Omega} \quad (26)$$

where,

$$B_h(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - p; \mathbf{v}_h, q_h) = B(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - p; \mathbf{v}_h, q_h) + S_h B(\tilde{p}_h - p, q_h).$$

The terms of  $B(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - p; \mathbf{v}_h, q_h)$  will be estimated using the approximation properties of the interpolations  $j_h$  and  $i_h$ ,

$$|\nu(\nabla(\tilde{\mathbf{u}}_h - \mathbf{u}), \nabla \mathbf{v}_h)| \leq \nu \|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \leq Ch^r \|\mathbf{u}\|_{r+1,\Omega} \|(\mathbf{v}_h, q_h)\| \quad (27)$$

and

$$|(p - \tilde{p}_h, \nabla \cdot \mathbf{v}_h)| \leq C \|p - \tilde{p}_h\|_{0,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \leq Ch^r \|p\|_{r,\Omega} \|(\mathbf{v}_h, q_h)\|. \quad (28)$$

To estimate the last term we shall use the integration by part and the orthogonality property of  $j_h$ ,

$$|(\nabla \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}), q_h)| = |(\tilde{\mathbf{u}}_h - \mathbf{u}, \nabla q_h)| = |(\tilde{\mathbf{u}}_h - \mathbf{u}, \kappa_h \nabla q_h)|. \quad (29)$$

Thus,

$$|(\nabla \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}), q_h)| \leq \left( \sum_{M \in M_h} \alpha_M^{-1} \|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{0,M}^2 \right)^{1/2} \left( \sum_{M \in M_h} \alpha_M \|\kappa_h \nabla q_h\|_{0,M}^2 \right)^{1/2}. \quad (30)$$

Hence,

$$\begin{aligned} |(\nabla \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}), q_h)| &\leq \left( \sum_{M \in M_h} C^2 \alpha_M^{-1} h_M^{2r+2} \|\mathbf{u}\|_{r+1,\omega(M)}^2 \right)^{1/2} \left( \sum_{M \in M_h} \alpha_M \|\kappa_h \nabla q_h\|_{0,M}^2 \right)^{1/2} \\ &\leq C \left( \sum_{M \in M_h} \alpha_M^{-1} h_M^{2r+2} \|\mathbf{u}\|_{r+1,\omega(M)}^2 \right)^{1/2} \|(\mathbf{v}_h, q_h)\|, \end{aligned} \quad (31)$$

i.e.

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$$\left| (\nabla \cdot (\tilde{\mathbf{u}}_h - \mathbf{u}), q_h) \right| \leq C^* h^r \|\mathbf{u}\|_{r+1, \Omega} \left\| (\mathbf{v}_h, q_h) \right\| \quad (32)$$

for some positive constant  $C^*$ .

Now, to estimate the stabilizing term  $S_h B(\tilde{p}_h - p, q_h)$  we use the  $L^2$ -stability of the fluctuation operator  $\kappa_h$  and the approximation property of  $i_h$ . We obtain the following:

$$\begin{aligned} |S_h B(\tilde{p}_h - p, q_h)| &= \left| \sum_{M \in M_h} \alpha_M (K_h \nabla(\tilde{p}_h - p), K_h \nabla P_h) \right| \\ &\leq \left( \sum_{M \in M_h} \alpha_M \|\kappa_h \nabla(\tilde{p}_h - p)\|_{0, M}^2 \right)^{1/2} \left( \sum_{M \in M_h} \alpha_M \|\kappa_h \nabla q_h\|_{0, M}^2 \right)^{1/2}. \end{aligned} \quad (33)$$

Thus by (11) we get

$$\begin{aligned} |S_h B(\tilde{p}_h - p, q_h)| &\leq \tilde{C} \left( \sum_{M \in M_h} \alpha_M h_M^{2r-2} |\nabla(\tilde{p}_h - p)|_{r-1, M}^2 \right)^{1/2} \left\| (\mathbf{v}_h, q_h) \right\| \\ |S_h B(\tilde{p}_h - p, q_h)| &\leq \tilde{C} \left( \sum_{M \in M_h} \alpha_M h_M^{2r-2} |\tilde{p}_h - p|_{r, M}^2 \right)^{1/2} \left\| (\mathbf{v}_h, q_h) \right\| \end{aligned}$$

using  $\alpha_M = Ch_M^2$ , we have

$$|S_h B(\tilde{p}_h - p, q_h)| \leq C_1 \left( \sum_{M \in M_h} h_M^{2r} \|p\|_{r, M}^2 \right)^{1/2} \left\| (\mathbf{v}_h, q_h) \right\|. \quad (34)$$

Hence

$$|S_h B(\tilde{p}_h - p, q_h)| \leq Ch^r \|p\|_{r, \Omega} \left\| (\mathbf{v}_h, q_h) \right\|. \quad (35)$$

Using (26) and (35), we get

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1, \Omega} + \|\tilde{p}_h - p\|_{0, \Omega} \leq Ch^r \left( \|\mathbf{u}\|_{r+1, \Omega} + \|p\|_{r, \Omega} \right). \quad \square$$

The required error estimate then follows using the triangular inequality together with the interpolant estimates. As already observed, the existence of the interpolation operators  $j_h$  and  $\mathbf{j}_h$  is fundamental, in *LPS* methods, for deriving the error estimates. Since, Assumption A3 may not be fulfilled, the existence of these operators is not always guaranteed.

### 3.3 Two-level local projection stabilization

This class of stabilized schemes uses discontinuous pressure gradient projection approximations of degree  $r-1$  which are defined on the coarser mesh  $M_h \equiv T_{2h}$ . Let  $\pi_{2h, r-1}$  be the  $L^2$ -projection onto  $D_h$  then  $k_h = id - \pi_{2h, r-1}$ . Here, for rectangular elements, the spaces  $Y_h$  and  $D_h$  are chosen as

$$Y_h / D_h = Q_{r, h} / Q_{r-1, 2h}^{disc}. \quad (36)$$

It has been proved in (Matthies *et al.*, 2007) that the pairs given by (36) satisfy Assumptions A3.

### 3.4 One-level enriched equal order stabilization

From the implementation point of view, a drawback of the two-level approach is the fact that the added stabilizing term produces a larger stencil which may not fit in the data structure of an available programming

code. A general theory which allows the derivation of a novel class of local projection stabilization by enrichment of the approximation spaces has been established in (Matthies *et al.*, 2007). This class of stabilized schemes uses approximation and projection spaces defined on the same mesh and leads to much more compact stencils than in the two-level approach.

Let  $M_h \equiv T_h$ ,  $\pi_{h,r-1}$  be the  $L^2$ -projection onto  $D_h$ , and  $\kappa_h \equiv id - \pi_{h,r-1}$ . For rectangular elements, the spaces  $Y_h$  and  $D_h$  are chosen as

$$Y_h / D_h = Q_{r,h}^{b,1} / Q_{r-1,h}^{disc} \quad (37)$$

or

$$Y_h / D_h = Q_{r,h}^{b,2} / P_{r-1,h}^{disc} \quad (38)$$

where

$$Q_{r,h}^{b,1}(\hat{K}) = Q_{r,h}(\hat{K}) \oplus \text{span}\{b_{\hat{K}} \hat{x}_i^{r-1}, i = 1, \dots, d\} \quad (39)$$

$$Q_{r,h}^{b,2}(\hat{K}) = Q_{r,h}(\hat{K}) + b_{\hat{K}} Q_{r-1,h}(\hat{K}) \quad (40)$$

and  $b_{\hat{K}}$  is a biquadratic bubble function on the reference element  $\hat{K}$ . Here also the pairs given by (37) and (38) satisfy Assumptions A3.

Note that, the local projection method described above does not add an extra cost to the solution of Stokes algebraic system. This is due to the elimination of pressure gradient unknowns at element level (Nafa and Wathen, 2009).

## 4. Numerical results

Numerical results for two Stokes problems are presented. The performance of the first order two-level method  $Q_{1,h} / Q_{0,2h}$  and the one-level method  $Q_{1,h}^b / Q_{0,h}$  approximations are obtained for  $\alpha_M = \alpha_0 h_M^2$  where  $\alpha_0$  is a constant.

### 4.1 Problem 1

The first problem consists of solving Stokes problem in the unit square with exact solution  $([u_1, u_2]^T, p)$  given by

$$u_1(x, y) = 2x^2(1-x)^2y(1-y)y(1-2y), \quad u_2(x, y) = -2x(1-x)(1-2x)y^2(1-y)^2$$

and

$$p(x, y) = x - x^2.$$

Numerical results are obtained, using the *LPS* method, for  $\nu = 1$ . Calculations were performed for  $\alpha_0 = 0.1$  and  $\alpha_0 = 1$  using the two-level and one-level methods.

The results given in Figures 2-5 indicate that the error norms  $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$  and  $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$  converge at the predicted rates, while  $\|p - p_h\|_{0,\Omega}$  seems to converge at a rate of about 3/2 as observed by Becker and Braack (2001) and by Nafa and Wathen (2009).

# LOCAL PROJECTION STABILIZATION METHOD

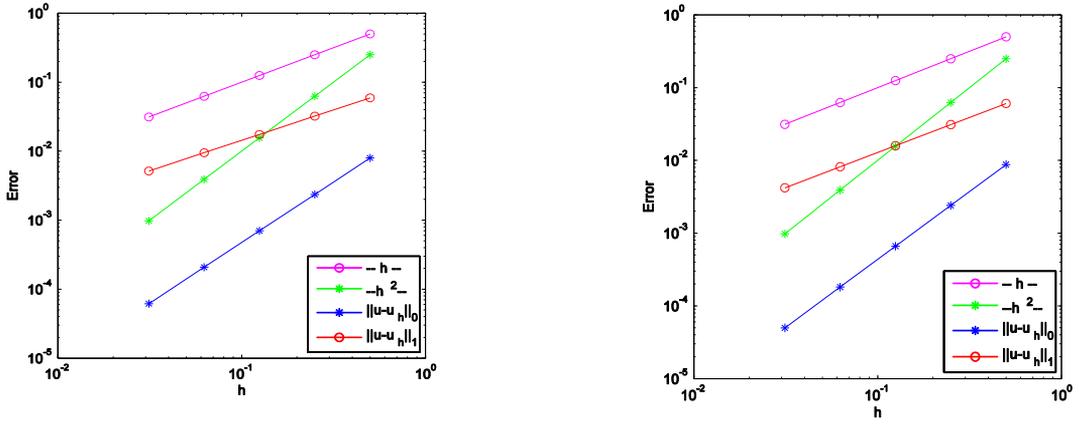


Figure 2. Velocity error norm using  $Q_{1,h}/Q_{0,2h}$  interpolation with  $\alpha_0 = 0.1$  (left) and  $\alpha_0 = 1$  (right).

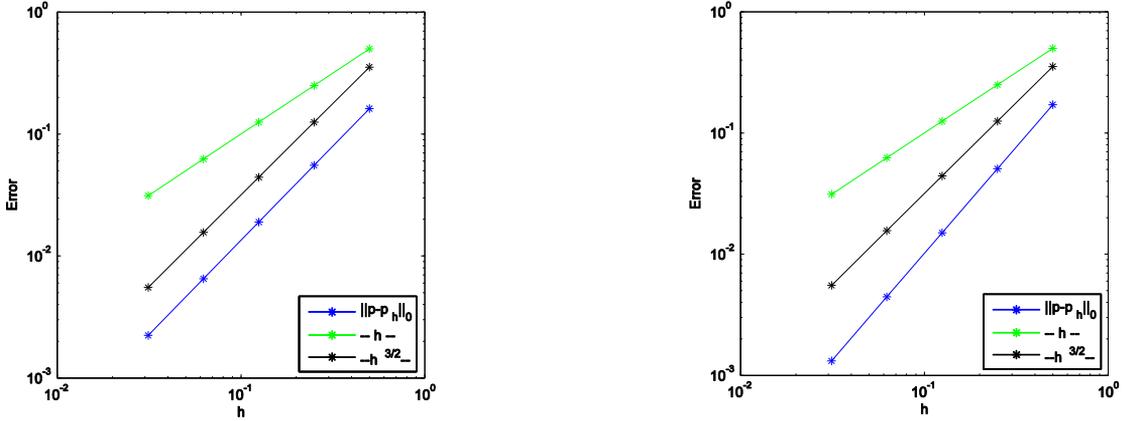


Figure 3. Pressure error norm using  $Q_{1,h}/Q_{0,2h}$  interpolation with  $\alpha_0 = 0.1$  (left) and  $\alpha_0 = 1$  (right).

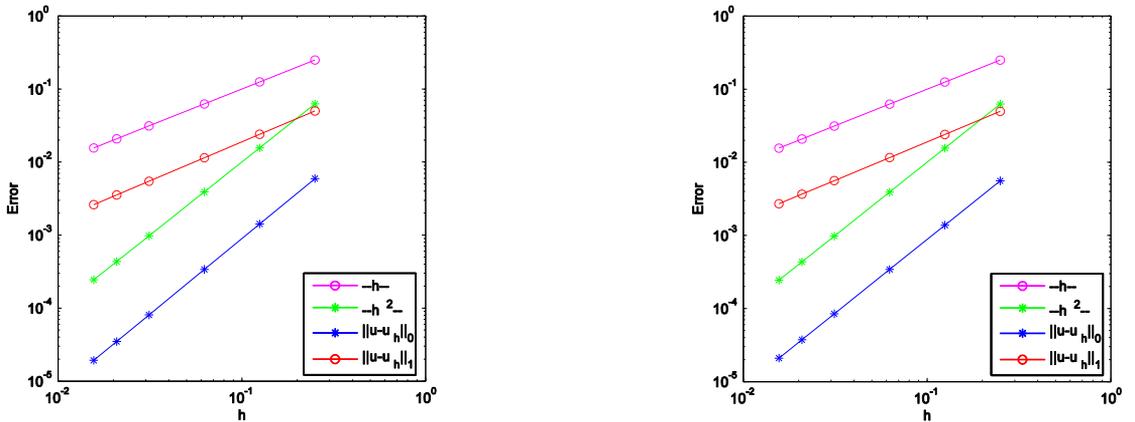


Figure 4. Velocity error norm using  $Q_{1,h}^b/Q_{0,h}$  interpolation with  $\alpha_0 = 0.1$  (left) and  $\alpha_0 = 1$  (right).

4.2 Problem 2

To test the accuracy of the method, we consider the standard Poisseuille flow in the channel  $\Omega=[0,4]\times[0,1]$ . We prescribe a parabolic inflow profile and use natural outlet conditions. The Dirichlet boundary conditions are not imposed on the whole boundary and natural boundary conditions

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p \mathbf{n} = 0$$

are employed. The solution of the problem is given by

$$u_1(x, y) = 1 - 4(y - 1/2)^2, \quad u_2(x, y) = 0 \quad \text{and} \quad p(x, y) = 8 - 2x.$$

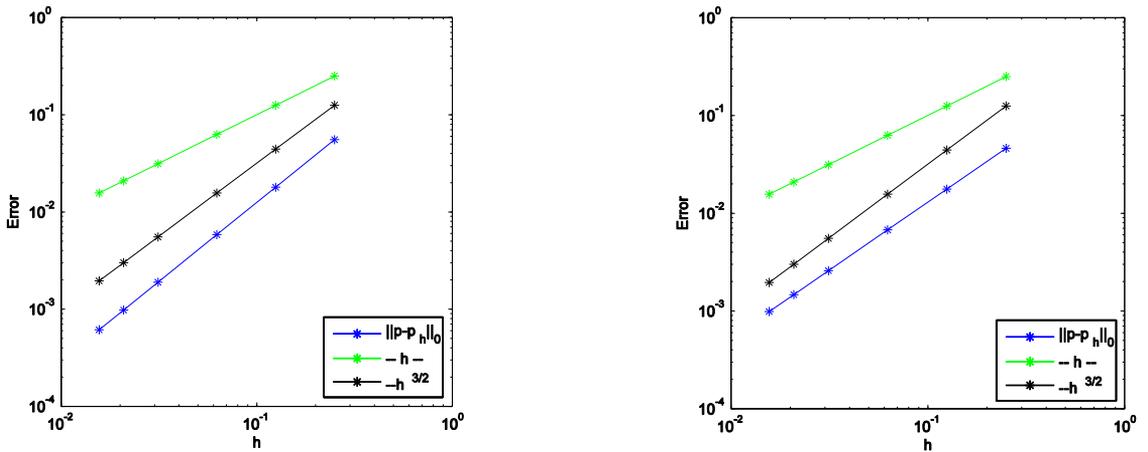


Figure 5. Pressure error norm using  $Q_{1,h}^b / Q_{0,h}$  interpolation with  $\alpha_0 = 0.1$  (left) and  $\alpha_0 = 1$  (right).

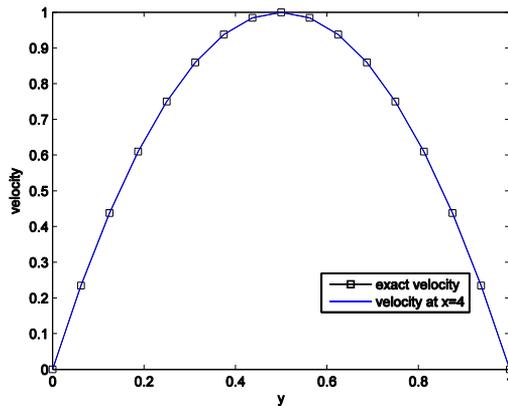


Figure 6. Velocity profile using  $Q_{1,h} / Q_{0,2h}$  interpolation with  $\alpha_0 = 0.1$ .

# LOCAL PROJECTION STABILIZATION METHOD

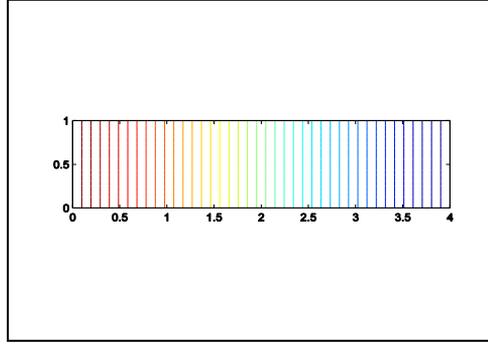


Figure 7. Pressure contours using  $Q_{1,h}/Q_{0,2h}$  interpolation with  $\alpha_0 = 0.1$ .

First, the problem is solved using  $Q_{1,h}/Q_{0,2h}$  approximation on a  $32 \times 16$  grid. The obtained results are illustrated in Figures 6 and 7 respectively, for the velocity and pressure solutions. We observe that the approximated pressure solution is exact even near the boundaries. In addition, Figure 8 shows that the velocity profile at the outlet boundary is in perfect agreement with the exact solution there.

The numerical solution of the Poiseuille problem is also performed on a  $32 \times 16$  grid using the one-level enriched approximation  $Q_{1,h}^b/Q_{0,h}$ . We obtain similar results to the two-level method for the velocity profiles as shown in Figure 9. The pressure solutions obtained for  $\alpha_0 = 10, 10^2, 10^3$ , and  $10^4$ , respectively, are illustrated in Figure 9. We note that while varying the coefficient  $\alpha_M = \alpha_0 h_M^2$  over the wide range from 0.1 to  $10^4$ , the obtained results change considerably. These results show that the pressure solution for the enriched one-level method is more sensitive to the choice of the parameter  $\alpha_0$  as compared to the two-level method.

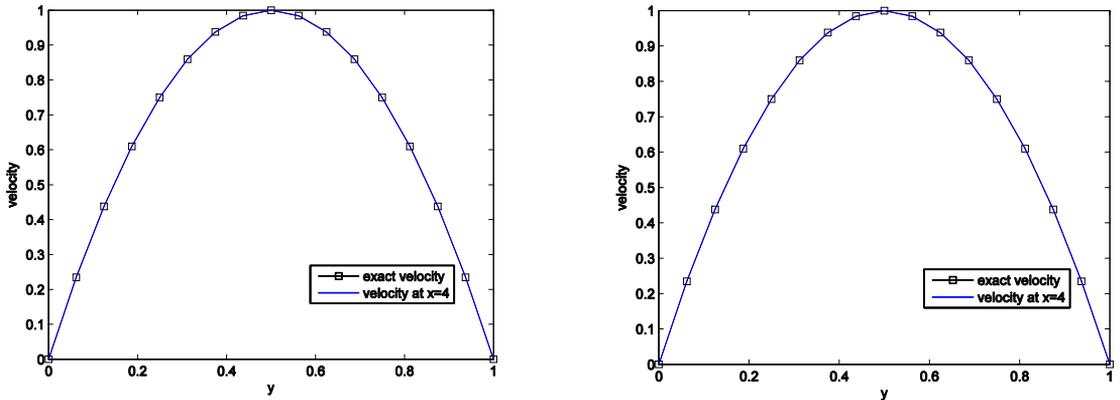


Figure 8. Velocity profile at the outlet of the channel using  $Q_{1,h}^b/Q_{0,h}$  interpolation with  $\alpha_0 = 0.1$  (left) and  $\alpha_0 = 1$  (right).

## 5. Conclusion

The convergence, stability and accuracy of *LPS* for the Stokes problem has been investigated using the one-level and two-level bilinear finite element interpolation. The numerical results show that the order of convergence for both approaches is as predicted theoretically for the velocity. But, the  $L^2$  rate of convergence for the pressure is approximately  $3/2$  as noted by other researchers (Becker and Braack, 2001; Nafa and Wathen, 2009). Also, we observe that the computation of the Poisseuille solution produces the exact velocity independently of the parameter  $\alpha_0$  for both methods. However, it is shown that the pressure is exact even near the boundaries for the two-level method and depends on the parameter  $\alpha_0$  for the one-level enriched method. In fact, to obtain results similar to those obtained by the two-level method, we have to take large values for  $\alpha_0$ .

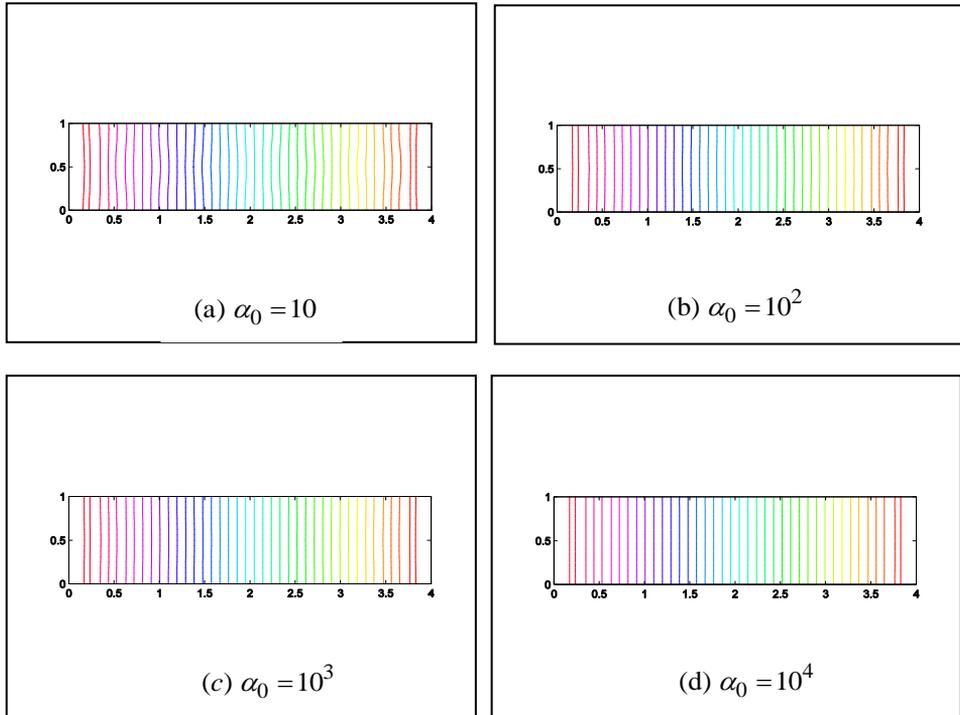


Figure 9. Pressure contours using  $Q_{1,h}^b / Q_{0,h}$  interpolation with different values of  $\alpha_0$ .

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