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A Galerkin Method for a Gaseous Ignition Model

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ABSTRACT: We consider a Galerkin procedure to solve a parabolic integrodifferential equation that arises in a gas combustion model. This model has been proposed by Kassoy and Poland, and subsequently analyzed by Bebernes, Eberly and Bressan. The problem is formulated in the variational form. In order to estimate the error, some intermediate projection has been employed. Under certain conditions on the given data, the L^2 error estimate has been obtained. A fully descretized version by using an extrapolated Crank-Nicolson method has been applied and the order of convergence derived.

KEYWORDS: Crank-Nicolson, Error estimate, Galerkin method, Gaseous ignition model.

طريقة جالاركين لنموذج إشعال غازي

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ملخص: نفترض طريقة جالاركين لحل المعادلة التفاضلية المكافئة التي تنشأ عن نموذج احتراق الغاز. لقد تم اقتراح هذا النموذج من قبل كاسي وبولاند وفيما بعد تم تحليله بواسطة بيبيرنيس وابيرلي و ريسان. وقد طرحت المشكلة في شكل متنوع. وبغية تقدير الخطأ، فقد استخدم إسقاط متوسط. تم الحصول على تقدير الخطأ L^2 تحت شروط مناسبة على بيانات معطية. وتم تطبيق أدلة الأبعاد التامة باستخدام طريقة كرانك-نيكلسون وإيجاد درجة التقارب.

1. Introduction

Kassoy and Poland (1983) developed an ignition model for a reactive gas in a bounded container to describe the induction period. During this period, the spacially uniform pressure increases and causes heating effects in the system. The pressure of the gas can be expressed in terms of a space integral term in the induction model that governs the temperature perturbation $u(x,t)$. This model is described by the set of equations (Bebernes and Bressan, 1988)

$$u_t - \Delta u = \delta e^u + \frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega u_t(x,t)dx, \quad (x,t)\in\Omega\times(0,\infty), \tag{1}$$

$$u(x,0) = g(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and volume $|\Omega|$, $\gamma > 1$.

The model has been subsequently studied by Bebernes and Bressan (1982), Bebernes and Bressan (1988), and Bebernes *et al*. (1989). Bebernes and Bressan (1982) analyzed this model and proved that for any positive value of the Frak-Kamenetski parameter δ and any value of the gas constant $\gamma \geq 1$, equations (1) have a unique classical solution $u(x,t)$ on $\Omega \times [0,T)$, where Ω is a bounded domain and T can be infinite. When T is finite, the solution blows up as $t \to T$. For a critical value δ_{crit} (see (Kassoy and Poland, 1983)), and $\delta > \delta_{\mathrm{crit}}$, the solution blows up in a finite time.

Bebernes and Eberly (1989) used the semigroup analysis to show the existence and uniqueness of a nonextended solution. Additional comparison results have been provided in the case of a spherically symmetric domain. Blowup occurs at a time $\sigma < T$ where T is the blowup time of the solid fuel ignition model. The location of the blowup has been also discussed. Depending on the nonlinearity of f, blowup can take place everywhere or at a single point (Bebernes and Eberly, 1989).

In this paper, we study a finite element approximation to the solution of the gas combustion model that is described by the partial differential equation (Bebernes and Eberly, 1989)

$$u_t - \Delta u = f(u) + \frac{\gamma - 1}{\gamma}\frac{1}{|\Omega|}\int_\Omega u_t(x,t)\,dx, \quad (x,t) \in \Omega \times (0,\infty), \tag{4}$$

$$u(x,0) = g(x), \quad x \in \Omega, \tag{5}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty). \tag{6}$$

We assume f is a Lipschitz function such that $f(u) > 0, f'(u) \geq 0$, and $f''(u) \geq 0$. In this work we develop estimates for error when a Galerkin method is applied. The error is optimal in the sense of the L^2 norm. This work is motivated by that of Cannon and Lin (1990a, 1990b). An extensive study of the finite element method for parabolic equations can be found in a book by Thomée (2006).

2. Formulation of the variational problem and Galerkin approximation

Let S_h be a finite dimensional subspace of the Sobolev space $H_0^1(\Omega)$ such that

$$\inf_{w \in S_h} (\|w - v\| + h\|\nabla(w - v)\|) \leq Ch^s \|v\|_s, \quad v \in H^s(\Omega) \cap H_0^1(\Omega), \tag{7}$$

where $s \geq 1$, $\|\cdot\|$ is the L^2 norm, and $\|\cdot\|_s$ is the Sobolev norm defined on $H^s(\Omega)$.

Problem (4) is equivalent to finding a $u \in H^s(\Omega) \cap H_0^1(\Omega)$ such that

$$(u_t, v) + (\nabla u, \nabla v) = (f(u), v) + \frac{\gamma - 1}{\gamma}\frac{1}{|\Omega|}\int_\Omega u_t\,dx \int_\Omega v\,dx, \quad \text{for all } v \in H_0^1(\Omega), \tag{8}$$

where $(\cdot,\cdot)$ is the inner product on $L^2(\Omega)$ defined as $(u,v) = \int_\Omega uv\,dx$.

The continuous Galerkin approximation $U : [0,T] \to S_h$ is defined as a solution to

$$(U_t, \chi) + (\nabla U, \nabla \chi) = (f(U), \chi) + \frac{\gamma - 1}{\gamma}\frac{1}{|\Omega|}\int_\Omega U_t\,dx \int_\Omega \chi\,dx, \quad \chi \in S_h$$

$$U(x,0) = G(x), \tag{9}$$

where $G(x)$ is the L^2 projection of $g(x)$ into S_h, i.e.,

$$(G-g,\chi)=0 \quad \text{for} \quad \chi\in S_h .$$

Given a basis $\{\phi_i\}_{i=1}^M$ for S_h, U can be written as

$$U(x,t)=\sum_{i=1}^{M}\alpha_i(t)\phi_i(x).$$

Then the variational equation can be written as the nonlinear initial value problem

$$B\mathbf{a}'(t)+A\mathbf{a}(t)=F(\mathbf{a}(t)), \quad C\mathbf{a}(0)=\mathbf{g}, \tag{10}$$

where A, B, and C are the matrices

$$A=(\nabla\phi_i,\nabla\phi_j),$$

$$B=(b_{ij})=(\phi_i,\phi_j)-\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\phi_i\,dx\ \int_\Omega\phi_i\,dx ,$$

$$C=(\phi_i,\phi_j),$$

for $i, j=1,2,\cdots,M$, and the vectors **a**, **F**, and **g** are defined by

$$\mathbf{a}(t)=(\alpha_i(t)),$$

$$\mathbf{F}(\mathbf{a})=(f(\sum_{i=1}^{M}\alpha_i\phi_i),\phi_j),$$

$$\mathbf{g}=(g,\phi_i).$$

The matrix B is positive definite, since

$$\begin{aligned}\mathbf{a}^T B\mathbf{a} &= \sum_{i=1}^{M} b_{ij}\alpha_i\alpha_j\\ &= \left\|\sum_{i=1}^{M}\alpha_i\phi_i\right\|^2-\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\Big(\int_\Omega\sum_{i=1}^{M}\alpha_i\phi_i\,dx\Big)^2\\ &= \|U\|^2-\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\Big(\int_\Omega U\,dx\Big)^2\\ &\ge \frac{1}{\gamma}\|U\|^2>0 \text{ for } U\neq 0,\end{aligned}$$

where we used the Schwarz inequality $(\int_\Omega U\,dx)^2\le\int_\Omega U^2dx\ \int_\Omega 1^2dx=|\Omega|\,\|U\|^2$.

With the assumption that $f(u)$ is uniformly Lipschitz, then it follows from the theory of ordinary differential equations that the initial-value problem (10) has a unique solution for $t>0$.

3. Projection of the solution

Let $W:[0,T]\to S_h$ such that

$$(\nabla(u-W),\nabla\chi)=0 \quad \text{for all} \quad \chi\in S_h . \tag{11}$$

Then W is the elliptic projection of $u\in H^s(\Omega)\cap H_0^1(\Omega)$ into S_h that satisfies the following properties (Thomée, 2006)

$$\|u-W\|\le Ch^s\|u\|_s , \tag{12}$$

$$\|\nabla u-\nabla W\|\le Ch^{s-1}\|u\|_s , \tag{13}$$

$$\|u_t-W_t\|\le Ch^s\|u\|_s . \tag{14}$$

4. Error estimates

Let $u-U=u-W+W-U=\eta+\theta$, where $\eta=u-W$ and $\theta=W-U$. From (8), (9) and (11), we get

$$\begin{aligned}(\theta_t,\chi)+(\nabla\theta,\nabla\chi)&=(W_t-U_t,\chi)+(\nabla W-\nabla U,\nabla\chi)\\&=(W_t,\chi)+(\nabla W,\nabla\chi)-(U_t,\chi)-(\nabla U,\nabla\chi)\\&=(W_t,\chi)+(\nabla u,\nabla\chi)-(f(U),\chi)-\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega U_t\,dx\int_\Omega\chi\,dx\\&=(W_t-u_t,\chi)+(f(u)-f(U),\chi)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega(u_t-U_t)\,dx\int_\Omega\chi\,dx\,,\end{aligned}$$

i.e.,

$$(\theta_t,\chi)+(\nabla\theta,\nabla\chi)=-(\eta_t,\chi)+(f(u)-f(U),\chi)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\eta_t\,dx\int_\Omega\chi\,dx+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\theta_t dx\int_\Omega\chi\,dx\,. \tag{15}$$

We choose $\chi=\theta$ and rewrite the equation

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2-\frac{1}{2}\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\frac{d}{dt}(\int_\Omega\theta\,dx)^2+\|\nabla\theta\|^2=-(\eta_t,\theta)+(f(u)-f(U),\theta)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\eta_t\,dx\int_\Omega\theta\,dx\,. \tag{16}$$

Assuming that f is uniformly Lipschitz with

$$|f(u_1)-f(u_2)|\le L|u_1-u_2|\,. \tag{17}$$

Then, using Schwarz and Young's inequalities implies

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2-\frac{1}{2}\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\frac{d}{dt}(\int_\Omega\theta dx)^2+\|\nabla\theta\|^2\le\frac{C}{\varepsilon}\|\eta_t\|^2+C\varepsilon\|\theta\|^2+L\|\theta\|^2+L\|\eta\|\|\theta\|\,. \tag{18}$$

With the use of Poincaré-Friedrichs' inequality (Gilbarg and Trudinger, 1983)

$$\|u\|\le\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\|\nabla u\|\ ,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2-\frac{1}{2}\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\frac{d}{dt}(\int_\Omega\theta dx)^2+(\frac{|\Omega|}{\omega_n})^{2/n}\|\theta\|^2\le\frac{C}{\varepsilon}\|\eta_t\|^2+\frac{C}{\varepsilon}\|\eta\|^2+(L+C\varepsilon)\|\theta\|^2\,. \tag{19}$$

If the Lipschitz constant of f is small enough such that

$$L<(\frac{|\Omega|}{\omega_n})^{2/n}\ , \tag{20}$$

then we can also choose ε small enough so that $(\frac{|\Omega|}{\omega_n})^{2/n}\ge L+C\varepsilon$. Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2-\frac{1}{2}\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\frac{d}{dt}(\int_\Omega\theta dx)^2+C\|\theta\|^2\le C\|\eta\|^2+C\|\eta_t\|^2\,. \tag{21}$$

Integrating both sides from 0 to t after dropping $C\|\theta\|^2$ to get

$$\|\theta\|^2-\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}(\int_\Omega\theta dx)^2\le\|\theta(\cdot,0)\|^2+C\int_0^t(\|\eta\|^2+\|\eta_t\|^2)\,d\tau\,.$$

Then, using schwarz inequality to obtain

$$\frac{1}{\gamma|\Omega|}\|\theta\|^2\le\|\theta(\cdot,0)\|^2+C\int_0^t(\|\eta\|^2+\|\eta_t\|^2)\,d\tau\,.$$

Here

$$\begin{aligned}\|\theta(\cdot,0)\|^2 &= \|W(\cdot,0)-U(\cdot,0)\|\\ &\le \|W(\cdot,0)-u(\cdot,0)\|+\|u(\cdot,0)-U(\cdot,0)\|\\ &\le Ch^s\|g\|_s+\|g-G\|\le Ch^s\|g\|_s\,,\end{aligned}$$

and $\|\eta\|^2+\|\eta_t\|^2$ can be replaced by their upper bound in (12) and (14). This implies

$$\|u-U\|\le Ch^s\|g\|_s+Ch^s\int_0^t(\|u\|_s+\|u_t\|_s)\,d\tau\,. \tag{22}$$

This establishes the following theorem.

Theorem 1. Suppose that problem (4) possesses a solution u in $H^s(\Omega)\cap H_0^1(\Omega)$, u_t in $H^s(\Omega)$, and f is uniformly Lipschitz that satisfies (17) and (20). Then, the continuous Galerkin solution U of (9) satisfies (22).

Proof. The next step is to get an estimate for $\nabla(u-W)$. For that purpose we put $\chi=\theta_t$ in (15). This yields

$$\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2=-(\eta_t,\theta_t)+(f(u)-f(U),\theta_t)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\eta_t\,dx\int_\Omega\theta_t\,dx+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\left(\int_\Omega\theta_t\,dx\right)^2. \tag{23}$$

This implies

$$\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2\le\frac{C}{\varepsilon}\|\eta_t\|^2+C\varepsilon\|\theta_t\|^2+\frac{C}{\varepsilon}\|f(u)-f(U)\|^2+\frac{\gamma-1}{\gamma}\|\theta_t\|^2. \tag{24}$$

Estimating the righthand side we get

$$\frac{1}{\gamma}\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2\le\frac{C}{\varepsilon}\|\eta_t\|^2+C\varepsilon\|\theta_t\|^2+\frac{CL^2}{\varepsilon}\|u-U\|^2. \tag{25}$$

Selecting ε small so that $\frac{1}{\gamma}>C\varepsilon$, we can drop the $\|\theta_t\|$ terms to get

$$\frac{d}{dt}\|\nabla\theta\|^2\le C\|\eta_t\|^2+C\|u-U\|^2. \tag{26}$$

Upon integrating from 0 to t, we get

$$\|\nabla\theta\|^2\le\|\nabla\theta(\cdot,0)\|^2+C\int_0^t\|\eta_t\|^2+C\int_0^t\|u-U\|^2)d\tau\,, \tag{27}$$

where

$$\begin{aligned}\|\nabla\theta(\cdot,0)\|&\le\|\nabla u(\cdot,0)-\nabla U(\cdot,0)\|+\|\nabla W(\cdot,0)-\nabla u(\cdot,0)\|\\ &\le\|\nabla g-\nabla G\|+Ch^{s-1}\|g\|_{s-1}\,,\end{aligned} \tag{28}$$

and

$$\int_0^t\|u-U\|\,d\tau\le\int_0^t\left(Ch^{s-1}\|g_{s-1}\|_{s-1}+\|g-G\|+Ch^{s-1}\int_0^\tau(\|u(\cdot,\beta)\|_s+\|u_t(\cdot,\beta)\|_s)\,d\beta\right)d\tau\,. \tag{29}$$

The double integration can be interchanged, a process to suppress one of the integrals, then the right hand side simplifies to

$$\int_0^t\|u-U\|\,d\tau\le C\int_0^t\Big[\|g-G\|+h^{s-1}(\|g\|_{s-1}+\|u\|_{s-1}+\|u_t\|_{s-1})\Big]d\tau\,. \tag{30}$$

In view of (28) and (30), estimate (27) may become

$$\|\nabla\theta\|^2\le\|\nabla g-\nabla G\|+Ch^{s-1}\|g\|_{s-1}+C\int_0^t\Big[\|g-G\|+h^{s-1}(\|g\|_{s-1}+\|u\|_{s-1}+\|u_t\|_{s-1})\Big]d\tau\,. \tag{31}$$

This proves the theorem.

Theorem 2. Under all the assumptions mentioned in Theorem 1, we have

$$\|\nabla u-\nabla U\|\le Ch^{s-1}\left\{\|g\|_{s-1}+\|u\|_{s-1}+\int_0^t\left(\|u\|_{s-1}+\|u_t\|_{s-1}\right)d\tau\right\}. \tag{32}$$

Note that as G being the L^2 projection of g onto S_h, , it legitimizes the estimates

$$\|g-G\|\le Ch^s\|g\|_s\ ,$$

$$\|\nabla g-\nabla G\|\le Ch^{s-1}\|g\|_{s-1}\,.$$

5. A priori estimate on extrapolated Crank-Nicolson-Galerkin method

In order to get a fully discretized version of the Galerkin method, we introduce the time mesh $t_m=mk$ for $m=0,1,\cdots,M$, where k is a uniform time step. For the rest of this section, we denote

$$\overline{F}_m=\frac{1}{2}(F_m+F_{m+1})\,,$$

as an averaged value of F on the nodes t_m and t_{m+1}.

In the Crank-Nicolson method, we replace the time derivative in (9) by $\partial U_m=(U_{m+1}-U_m)/k$ and U by $\overline{U}_m=(U_m+U_{m+1})/2$. This defines U_{m+1} as a solution to the nonlinear system

$$(\partial U_m,\chi)+(\nabla\overline{U}_m,\nabla\chi)=(f(\overline{U}_m),\chi)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\partial U_m\,dx\int_\Omega\chi\,dx\,,\qquad \chi\in S_h\,.$$

The nonlinearity due to $f(\overline{U}_m)$ can be overcome by replacing the argument of $f(\overline{U}_m)$ by an extrapolated U over the time steps m and $m-1$, i.e.

$$f(\overline{U}_m)\approx f\left(\frac{3}{2}U_m-\frac{1}{2}U_{m-1}\right).$$

We denote these extrapolated values by

$$\hat{F}_m=\frac{3}{2}F_m-\frac{1}{2}F_{m-1}\,. \tag{33}$$

This produces the new linearized equation in U_{m+1} as

$$(\partial U_m,\chi)+(\nabla\overline{U}_m,\nabla\chi)=(f(\hat{U}_m),\chi)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\partial U_m\,dx\int_\Omega\chi\,dx\,,\qquad \chi\in S_h\,. \tag{34}$$

Note that this extrapolation process will result in a second order accuracy

$$\hat{u}_m=\frac{3}{2}u_m-\frac{1}{2}u_{m-1}=u_{m+1/2}+O(k^2)\,,$$

with $u_{m+1/2}=u(.,t_{m+1/2})$. We shall estimate the error $\|U_m-u(\cdot,t_m)\|$

$$U_m-u(\cdot,t_m)=U_m-W(\cdot,t_m)+W(\cdot,t_m)-u(\cdot,t_m)=\theta_m+\eta_m\ ,$$

where the estimate of η_m is shown in (12). We now consider θ_m by writing

$$\begin{aligned}(\partial\theta_m,\chi)+(\nabla\overline{\theta}_m,\nabla\chi)&=(\partial U_m,\chi)+(\nabla\overline{U}_m,\nabla\chi)-(\partial W_m,\chi)+(\nabla\overline{W}_m,\nabla\chi)\\&=(f(\hat{U}_m)-\overline{f}(u_m),\chi)+\frac{\gamma-1}{\gamma}\frac{1}{|\Omega|}\int_\Omega\left(\partial U_m-\frac{\partial\overline{u}_m}{\partial t}\right)dx\int_\Omega\chi\,dx+\left(\frac{\partial\overline{u}_m}{\partial t}-\partial W_m,\chi\right),\end{aligned} \tag{35}$$

where

$$\bar{f}(u_m) = \frac{1}{2}(f(u_m) + f(u_{m+1})).$$

This implies

$$(\partial\theta_m, \chi) - \frac{\gamma - 1}{\gamma}\frac{1}{|\Omega|}\int_\Omega \partial\theta_m \, dx \int_\Omega \chi \, dx + (\nabla\bar{\theta}_m, \nabla\chi)$$

$$= (f(\hat{U}_m) - \bar{f}(u_m), \chi) + \frac{\gamma - 1}{\gamma}\frac{1}{|\Omega|}\int_\Omega \left(\partial\eta_m + \partial u_m - \frac{\partial\bar{u}_m}{\partial t}\right) dx \int_\Omega \chi \, dx - \left(\partial\eta_m + \partial u_m - \frac{\partial\bar{u}_m}{\partial t}, \chi\right). \quad (36)$$

Setting $\chi = \bar{\theta}_m$, we can get

$$\frac{1}{2}\partial\|\theta_m\|^2 - \frac{\gamma - 1}{2\gamma}\frac{1}{|\Omega|}\partial\left(\int_\Omega \theta_m \, dx\right)^2 + \|\nabla\bar{\theta}_m\|^2 \le C\left(\|f(\hat{U}_m) - \bar{f}(u_m)\| + \|\partial\eta_m\| + \left\|\partial u_m - \frac{\partial\bar{u}_m}{\partial t}\right\|\right)\|\nabla\bar{\theta}_m\|, \quad (37)$$

where we have used the Poincaré-Friedrichs' inequality $\|\bar{\theta}_m\| \le \|\nabla\bar{\theta}_m\|$. This implies

$$\frac{1}{2}\partial\|\theta_m\|^2 - \frac{\gamma - 1}{2\gamma}\frac{1}{|\Omega|}\partial\left(\int_\Omega \partial\theta_m \, dx\right)^2 \le C\left(\|f(\hat{U}_m) - \bar{f}(u_m)\|^2 + \|\partial\eta_m\|^2 + \left\|\partial u_m - \frac{\partial\bar{u}_m}{\partial t}\right\|^2\right). \quad (38)$$

The last two norms on the right hand side are of orders h^{2s} and k^4, respectively. Moreover

$$\begin{aligned} \|f(\hat{U}_m) - \bar{f}(u_m)\| &\le \|f(\hat{U}_m) - f(u_{m+1/2})\| + \|f(u_{m+1/2}) - \bar{f}(u_m)\| \\ &\le C(\|\hat{U}_m - u_{m+1/2}\| + k^2) \\ &\le C(\|\hat{\theta}_m\| + \|\hat{\eta}_m\| + \|\hat{u}_m - u_{m+1/2}\| + k^2) \\ &\le C(\|\theta_m\| + \|\theta_{m-1}\| + h^s + k^2), \end{aligned} \quad (39)$$

where $\hat{\theta}_m$, $\hat{\eta}_m$ and $\hat{u}_m$ are the extrapolated representations for θ_m, η_m and u_m, respectively. On the other hand, the left hand side of (38) is bounded below by

$$\frac{1}{2}\partial\|\theta_m\|^2 - \frac{\gamma - 1}{2\gamma}\frac{1}{|\Omega|}\partial\left(\int_\Omega \theta_m \, dx\right)^2 \ge \frac{1}{2\gamma}\partial\|\theta_m\|^2. \quad (40)$$

Now, in view of (39) and (40), estimate (38) can be written as

$$\|\theta_{m+1}\|^2 \le (1 + Ck)\|\theta_m\|^2 + Ck\|\theta_{m-1}\|^2 + Ck(h^s + k^2)^2,$$

or

$$\|\theta_{m+1}\|^2 + Ck\|\theta_m\|^2 \le (1 + 2Ck)(\|\theta_m\|^2 + Ck\|\theta_{m-1}\|^2) + Ck(h^s + k^2)^2. \quad (41)$$

A repeated application of (41), with a small *k*, implies

$$\|\theta_m\|^2 \le C(\|\theta_1\|^2 + k\|\theta_0\|^2) + C(h^s + k^2)^2.$$

If θ_0 and θ_1 are both calculated with an accuracy $O(h^s) + O(k^2)$, we get the following result

$$\|\theta_m\| \le C(h^s + k^2),$$

which proves the following theorem.

Theorem 3. The extrapolated Crank-Nicolson solution U_m of (34) satisfies

$$\max_m \|u_m - U_m\| \le C(h^s + k^2),$$

where *C* depends on *u*.

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