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Dihedral Groups as Epimorphic Images of Some Fibonacci Groups

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ABSTRACT: The Fibonacci groups are defined by the presentation $F\,(r,n) = \langle a_1, a_2, \ldots, a_n : a_1 a_2 \cdots a_r = a_{r+1}, a_2 a_3 \cdots a_{r+1} = a_{r+2}, \ldots,\ a_n a_1 \cdots a_{r-1} = a_r \rangle$, where $r > 0$, $n > 0$ and all subscripts are assumed to be reduced modulo n. In this paper we give an alternative proof that for $r \geq 0$, $F(2r, 4r+2)$, $F(4r+3, 8r+8)$ and $F(4r+5, 8r+12)$ are all infinite by establishing a morphism (or group homomorphism) onto the dihedral group D_n for all $n > 2$.[1]

Keywords: Group; Fibonacci group; Dihedral group; (homo) Morphism.

مجموعات دايهيدرل كصورة متماثلة لمجموعات فيبوناتشى

عبدالله عمر و بشير علي

ملخص : تعرّف مجموعات فيبوناتشي تعرف بواسطة التمثيل

$$F(r,n) = \langle a_1, a_2, ..., a_n : a_1, a_2, a_3 ...\ \ a_{r+1} = a_{r+2}, ..., a_n a_1 ... a_{r-1} = a_r \rangle$$

عندما تكون $n > 0, r > 0$ ، $a_r = a_{r+n}$ لكل $r = 1, ..., n$.
نعطي في هذا البحث برهانا بديلاً ، بأن $F(2r, 4r+2)$ ، $F(4r+3, 8r+8)$ و $F(4r+5, 8r+12)$ جمعيها لا منتهية في حالة $r \geq 0$ وذلك بواسطة إيجاد دالة زمرة متماثلة وفوقيه على الزمرة D_n لكل $n > 2$.

مفتاح الكلمات : مجموعات ، مجموعات فيبوناتشي ، مجموعات دايهيدرل ، تشابه شكلي.

1. Introduction

For $r \geq 1$ and $n \geq 1$ the Fibonacci group $F(r,n)$ is defined by the presentation:

$$F(r,n) = \langle a_1, a_2, \ldots, a_n : a_1 a_2 \cdots a_r = a_{r+1}, a_2 a_3 \cdots a_{r+1} = a_{r+2}, \cdots,\ a_n a_1 \cdots a_{r-1} = a_r \rangle,$$

where all subscripts are assumed to be reduced modulo n, if necessary. These groups were first introduced by Conway (1965) and have been studied over the last few decades. For a nice survey article see (Thomas, 1991) or (Campbell *et al*., 1992).

The dihedral group of order $2n$ denoted by D_n is usually defined by

$$D_n = \langle x, y : x^n = y^2 = 1, yx^{-1} = xy \rangle. \tag{1}$$

It is well known that x and y in D_n satisfy the relations summarized in the next lemma.

Lemma 1.1 For all $0 \leq k \leq n-1$ we have

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(a) $x^{-k} = x^{n-k}$;
(b) $y^{-1} = y$;
(c) $yx^{k} = x^{n-k}y$;
(d) $(x^{k}y)^{2} = 1$;
(e) $x^{k}yx^{k} = y$;
(f) $yx^{k}y = x^{n-k}$.

Thus we may write the elements of D_n uniquely as x^k or $x^k y$ for $k = 0, 1, 2, \dots, n-1$.

Campbell *et al.* (2004) explored the connection between the Fibonacci groups and finite groups *via* the concept of Fibonacci length. In the case where the finite groups were dihedral they obtained satisfactory results. In this note we further explore the connection between the Fibonacci groups and dihedral groups in a different manner. In particular, we establish epimorphisms between Fibonacci groups in certain classes and all finite dihedral groups of order greater than 4, thus giving alternative proofs regarding the infiniteness of the groups in these classes of Fibonacci groups. For basic concepts in group theory we refer the reader to (Gallian, 1998). The following lemma for $F(r,n)$ is indispensable for our discourse.

Lemma 1.2 For all $r > 0$ and $m \geq 2$ we have $a_{m+r} = a_{m-1}^{-1}a_{m+r-1}^{2}$ in $F(r,n)$.
Proof.

$$\begin{aligned} a_{m+r} &= a_m a_{m+1} \cdots a_{m+r-1} = a_{m-1}^{-1}(a_{m-1}a_m a_{m+1} \cdots a_{m+r-2})a_{m+r-1} \\ &= a_{m-1}^{-1}a_{m+r-1}^{2}. \end{aligned}$$

2. Morphic Images

First we consider the Fibonacci groups $F(2r, 4r+2)$.

Theorem 2.1 Let $r > 0$. There exist morphisms from $F(2r,\ 4r+2)$ onto D_n for all $n \geq 3$. Hence $F(2r, 4r+2)$ is infinite.

We are going to prove this theorem *via* a sequence of lemmas. However, we first define a mapping from the first $2r$ generators of $F(2r,\ 4r+2)$ onto the generators of D_n by

$$a_x \mapsto x \text{ and } a_i \mapsto y \ (i = 2, 3, \dots, 2r). \tag{2}$$

Then the next lemma gives the images of the remaining generators: $a_{2r+1}, a_{2r+2}, \cdots, a_{4r+1}$.

Lemma 2.2

(a) $a_{2r+1} \mapsto xy \ (r \geq 1)$;
(b) $a_{2r+2} \mapsto x^{n-1} (r \geq 1)$;
(c) $a_{2r+3} \mapsto x^{2}y \ (r \geq 1)$;
(d) $a_{2r+i} \mapsto y \ (r \geq 2$ and $4 \leq i \leq 2r+1)$.

Proof. Using Lemma 1.2 we see that
(a) $a_{2r+1} = a_1 a_2 \cdots a_{2r} \mapsto xy^{2r-1} = xy \ (r \geq 1)$;
(b) $a_{2r+2} = a_1^{-1}a_{2r+1}^{2} \mapsto x^{-1}(xy)^{2} = x^{n-1} (r \geq 1)$;
(c) $a_{2r+3} = a_2^{-1}a_{2r+2}^{2} \mapsto y^{-1}(x^{n-1})^{2} = x^{2}y \ (r \geq 1)$;
(d) This proof is by induction.

Basis step: By Lemma 1.2 and (c) above, we see that

$$a_{2r+4} = a_3^{-1}a_{2r+3}^{2} \mapsto y^{-1}(x^{2}y)^{2} = y.$$

Inductive step: Suppose that $a_{2r+i} \mapsto y$ (for some $4 \leq i \leq 2r$). Using Lemma 1.2 again we see that

$$a_{2r+i+1} = a_i^{-1}a_{2r+i}^{2} \mapsto y^{-1}y^{2} = y,$$

as required.
Lemma 2.3 For $r \geq 1$ we have

(a) $a_{4r+2} \mapsto xy\,(r \geq 1)$;
(b) $a_1 = a_{4r+3} \mapsto x$;
(c) $a_{i-2} = a_{4r+i} \mapsto y\ (4 \leq i \leq 2r+2)$.

Proof. Using Lemmas 1.2 and 2.2 we see that
(a) $a_{4r+2} = a_{2r+1}^{-1} a_{4r+1}^{2} \mapsto (xy)^{-1} y^2 = xy$;
(b) $a_{4r+3} = a_{2r+2}^{-1} a_{4r+2}^{2} \mapsto x(xy)^2 = x$;
(c) This proof is by induction.

Basis step: By Lemma 1.2 and (b) above, we see that

$$a_{4r+4} = a_{2r+3}^{-1} a_{4r+3}^{2} \mapsto (x^2 y)^{-1} x^2 = y.$$

Inductive step: Suppose that $a_{4r+i} \mapsto y$ (for some $4 \leq i \leq 2r+1$). Using Lemma 1.2 again we see that

$$a_{4r+i+1} = a_{2r+i}^{-1} a_{4r+i}^{2} \mapsto y^{-1} y^2 = y,$$

as required.

It is now clear from Lemmas 2.2 and 2.3 that the mapping defined in (2) is indeed a morphism onto D_n, which preserves all the relations of $F(2r,\ 4r+2)$ and so Theorem 2.1 is proved.

Next we consider the Fibonacci groups $F(4r+3, 8r+8)$.

Theorem 2.4 Let $r \geq 0$. There exist morphisms from $F(4r+3, 8r+8)$ onto D_n for all $n \geq 3$. Hence $F(4r+3,\ 8r+8)$ is infinite.

As in the previous case, we are going to prove this theorem *via* a sequence of lemmas. First, we define a mapping from the first $4r+3$ generators of $F(4r+3,\ 8r+8)$ onto the generators of D_n by

$$a_i, a_{2r+3} \mapsto x \ \text{ and } \ a_i \mapsto y, \tag{3}$$

where $2 \leq i \leq 4r+3, i \neq 2r+3$ and $r \geq 0$. Then the next two lemmas give the images of the remaining generators: $a_{4r+4}, a_{4r+5}, \cdots, a_{8r+8}$.

Lemma 2.5 For $r \geq 0$ we have
(a) $a_{4r+4} \mapsto y$;
(b) $a_{4r+5} \mapsto x^{n-1}$;
(c) $a_{4r+6} \mapsto x^2 y$;
(d) $a_{4r+i} \mapsto y\ (7 \leq i \leq 2r+6)$.

***Proof*.** Using Lemma 1.2 we see that
(a) $a_{4r+4} = a_1 a_2 \cdots a_{4r+3} \mapsto xy^{2r+1} xy^{2r} = y$;
(b) $a_{4r+5} = a_1^{-1} a_{4r+4}^{2} \mapsto x^{-1} y^2 = x^{n-1}$;
(c) $a_{4r+6} = a_2^{-1} a_{4r+5}^{2} \mapsto y^{-1} (x^{n-1})^2 = x^2 y$;
(d) This proof is by induction.

Basis step: By Lemma 1.2 and (c) above, we see that

$$a_{4r+7} = a_3^{-1} a_{4r+6}^{2} \mapsto y^{-1} (x^2 y)^2 = y.$$

Inductive step: Suppose that $a_{4r+i} \mapsto y$ (for some $7 \leq i \leq 2r+5$). Using Lemma 1.2 and the induction

hypothesis we see that
$$a_{4r+i+1} = a_{i-3}^{-1} a_{4r+i}^2 \mapsto y^{-1} y^2 = y,$$
as required.

Lemma 2.6 For $r \geq 0$ we have
(a) $a_{6r+7} \mapsto x^{n-1}$;
(b) $a_{6r+8} \mapsto x^2 y$;
(c) $a_{6r+i} \mapsto y$ $(9 \leq i \leq 2r+8)$.

Proof. Using Lemmas 1.2 and 2.5 we see that
(a) $a_{6r+7} = a_{2r+3}^{-1} a_{6r+6}^2 \mapsto x^{-1} y^2 = x^{n-1}$;
(b) $a_{6r+8} = a_{2r+4}^{-1} a_{6r+7}^2 \mapsto y^{-1}(x^{n-1})^2 = x^2 y$;
(c) This proof is by induction.

Basis step: For $i = 9$, we see tha
$$a_{6r+9} = a_{2r+5}^{-1} a_{6r+8}^2 \mapsto y^{-1}(x^2 y)^2 = y.$$
Inductive step: Suppose that $a_{6r+i} \mapsto y$ (for some $9 \leq i \leq 2r+7$). Then using Lemma 1.2, the fact that $i \geq 9$ and induction hypothesis we see that
$$a_{6r+i+1} = a_{(2r-3)+i}^{-1} a_{6r+i}^2 \mapsto y^{-1} y^2 = y,$$
as required.

Lemma 2.7 for $r \geq 0$ we have
(a) $a_1 = a_{8r+9} \mapsto x$;
(b) $a_{i-8} = a_{8r+i} \mapsto y$ $(10 \leq i \leq 2r+10)$.

Proof. Using Lemmas 1.2, 2.5 and 2.6 we see that
(a) $a_{8r+9} = a_{4r+5}^{-1} a_{8r+8}^2 \mapsto (x^{n-1})^{-1} y^2 = x$;
(b) for $10 \leq i \leq 2r+10$, we use induction.

Basis step: For $i = 10$, we see that
$$a_{(8r+8)+2} = a_{4r+6}^{-1} a_{(8r+8)+1}^2 \mapsto (x^2 y)^{-1} x^2 = y.$$
Inductive step: Suppose that $a_{8r+i} \mapsto y$ (for some $10 \leq i \leq 2r+9$). Then using Lemma 1.2, (a) above and the induction hypothesis we see that
$$a_{8r+i+1} = a_{4r+i-3}^{-1} a_{8r+i}^2 \mapsto y^{-1} y^2 = y,$$
as required.

Lemma 2.8 For $r \geq 0$ we have
(a) $a_{2r+3} = a_{8r+(2r+11)} \mapsto x$;
(b) $a_{i-8} = a_{8r+i} \mapsto y$ $(2r+12 \leq i \leq 4r+11)$.

Proof. Using Lemmas 1.2, 2.6 and 2.7 we see that
(a) $a_{8r+(2r+11)} = a_{4r+(2r+7)}^{-1} a_{8r+(2r+10)}^2 \mapsto (x^{n-1})^{-1} y^2 = x$;
(b) for $2r+12 \leq i \leq 4r+11$ we use induction.

Basis step: For $i = 2r+12$, we see that
$$a_{8r+(2r+12)} = a_{4r+(2r+8)}^{-1} a_{8r+(2r+11)}^2 \mapsto (x^2 y)^{-1} x^2 = y.$$

Inductive step: Suppose that $a_{8r+i} \mapsto y$ (for some $2r+12 \le i \le 4r+10$). Then using Lemma 1.2 (a) above and the induction hypothesis we see that

$$a_{8r+i+1} = a_{4r+i-3}^{-1} a_{8r+i}^{2} \mapsto y^{-1}y^{2} = y,$$

as required.

It is now clear from Lemmas 2.5, 2.6, 2.7 and 2.8 that the mapping defined in (3) is indeed a morphism onto D_n, which preserves all the relations of $F(4r+3, 8r+8)$ and so Theorem 2.4 is proved.

Finally we consider the Fibonacci groups $F(4r+5, 8r+12)$.

Theorem 2.9 Let $r \ge 0$. There exist morphisms from $F(4r+5, 8r+12)$ onto D_n for all $n \ge 3$. Hence $F(4r+5, 8r+12)$ is infinite.

As in the previous cases, we are going to prove this theorem *via* a sequence of lemmas. However, since the proofs are similar to the previous case we are going to state the corresponding results without proofs. We first define a mapping from the first $4r+5$ generators of $F\left(4r+5, 8r+12\right)$ onto the generators of D_n by

$$a_1, a_{2r+3} \mapsto x \text{ and } a_i \mapsto y, \tag{4}$$

where $2 \le i \le 4r+5, i \ne 2r+3$ and $r \ge 0$. Analogously to Lemma 2.5 we have

Lemma 2.10 For $r \ge 0$

(a) $a_{4r+6} \mapsto y;$

(b) $a_{4r+7} \mapsto x^{n-1};$

(c) $a_{4r+8} \mapsto x^{2}y;$

(d) $a_{4r+i} \mapsto y\ (9 \le i \le 2r+8).$

Analogously to Lemma 2.6 we have

Lemma 2.11 For $r \ge 0$ we have.

(a) $a_{6r+9} \mapsto x^{n-1};$

(b) $a_{6r+10} \mapsto x^{2}y;$

(c) $a_{6r+i} \mapsto y\ (11 \le i \le 2r+12).$

Analogously to Lemma 2.7 we have

Lemma 2.12 For $r \ge 0$ we have

(a) $a_1 = a_{8r+13} \mapsto x;$

(b) $a_{i-12} = a_{8r+i} \mapsto y\ (14 \le i \le 2r+14).$

Analogously to Lemma 2.8 we have

Lemma 2.13 For $r \ge 0$ we have

(a) $a_{2r+3} = a_{8r+(2r+15)} \mapsto x;$

(b) $a_{i-12} = a_{8r+i} \mapsto y\ (2r+16 \le i \le 4r+17).$

It is now clear from Lemmas 2.10, 2.11, 2.12 and 2.13 that the mapping defined in (4) is indeed a morphism onto D_n, which preserves all the relations of $F(4r+5, 8r+12)$ and so Theorem 2.9 is proved.

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