SQU Journal for Science, 2020, 25(2), 100-106 DOI:10.24200/squjs.vol25iss2pp100-106
Sultan Qaboos University

Doubly Periodic Functions and Floquet Theorem

Nafya H. Mohammed[1]* and Nazaneen Q.M. Saeed[2]

[1]Mathematics Department, College of Basic Education, University of Raparin, Kurdistan Region-Iraq. [2]Mathematics Department, College of Education, University of Salahaddin, Kurdistan Region-Iraq. *E-mail: nafya.mohammad@uor.edu.krd

ABSTRACT: In complex analysis, an elliptic function is a meromorphic function that is periodic in two directions. Just as a periodic function of a real variable is defined by its values on an interval, an elliptic function is determined by its values on a fundamental parallelogram, which then repeat in a lattice. Such a doubly periodic function cannot be holomorphic, as it would then be a bounded entire function, and by Liouville's theorem every such function must be constant. Historically, elliptic functions were first discovered by Niels Henrik Abel as inverse functions of elliptic integrals, and their theory was improved by Carl Gustav Jacobi; these in turn were studied in connection with the problem of the arc length of an ellipse, whence the name derives. In this paper, we extend Floquet theorem and another theorem (which is mentioned in [1]) related to it, which are dependent on elliptic functions.

Keywords: Meromorphic function; Periodic function; Elliptic function; Floquet Theorem; Fundamental matrix.

الدوال مضاعفة الدوران و مبرهنة فلوكت

نفيا حميد محمد و نازنين قادر محمد سعيد

الملخص: تعتبر الدالة الناقصة في التحليل العقدي دالة مرومورفية ذات دوران باتجاهين. مثلما يتم تعريف الدالة الدورية لمتحول حقيقي بقيمها على مجال، يتم تعريف الدالة الناقصة بقيمها على متوازي الأضلاع الأساسي، الذي يتكرر بعد ذلك في الشبكة. لا يمكن لهذه الدالة مضاعفة الدوران أن تكون تحليلية، لأنها ستكون عندئذ دالة مقيدة كليا، وحسب مبرهنة ليوفيل، فإن كل دالة يجب أن تكون ثابتة. تاريخيا، تم اكتشاف الدوال الناقصة لأول مرة بواسطة نيلزهنريك أبيل كدوال عكسية للتكاملات الناقصة، وتم تحسينها نظريا بواسطة كارل جوستاف جاكوبي؛ ومن ناحية أخرى تم دراسة هذه الدوال لعلاقتها مع مسألة طول القوس لقطع ناقص، ومن هذا تم إعطائها الإسم. نعمم في هذا البحث، مبرهنة فلوكيت و مبرهنة أخرى (المذكورة في [1]) ذات صلة بها، والتي تعتمد على الدوال الناقصة.

الكلمات المفتاحية: الدالة المرومورفية، الدالة الدورية، الدالة الناقصة، مبرهنة فلوكت، المصفوفة الأساسية.

1. Introduction

In our opinion, complex analysis is one of the most beautiful areas of mathematics. It has one of the highest ratios of theorems to definitions (i.e., a very low "entropy"), and many applications to things that seem unrelated to complex numbers. Also, it is a comprehensive subject, which provides every mathematician with helpful data. In this respect and due to the usefulness of this subject, we have chosen elliptic functions to be the focus of our work. We need to give a definition of what an elliptic function is, so we will restrict ourselves to meromorphic functions which are functions having only poles as singularities. A doubly periodic function is a function that has two primitive periods, namely $2w_1$ and $2w_3$ with

$$f(z + 2mw_1 + 2nw_3) = f(z); \quad m, n \in \mathbb{Z}.$$

The set of all points of the form $2mw_1 + 2nw_3$, with m and n being integers is called the period lattice. An elliptic function is a meromorphic function that admits two independent primitive periods. At least one of the two primitive periods of an elliptic function should be complex since the ratio of these two periods should be non-real.

In 1998, Gesztesy and Weikard [2] provided an overview of elliptic algebro-geometric solutions of the KdV and AKNS hierarchies, in which they concentrated on Floquet theorem. Also, Weikard in 2000 [3] dealt with differential equations with meromorphic solutions, which is related to Floquet theorem, while Chouikha [4] paid more attention to properties and developments of elliptic functions, and in particular Jacobi elliptic functions. The main work in this paper is the extension of the Floquet theorem based on elliptic functions.

2. Preliminaries

Definition 2.1. A function $f: \mathbb{C} \longrightarrow \mathbb{C}_\infty$ with two periods $2w_1$ and $2w_3$, the ratio of which is not real, is called 'doubly periodic'.
Definition 2.2. A function that is analytic in the region D except for poles in D, is called 'meromorphic' in D.
Definition 2.3. A doubly periodic meromorphic function is called 'elliptic'.

Table (1) contains 12 Jacobi elliptic functions (as examples of elliptic functions) with their periods, zeros, poles, and residues of the functions at the poles.

Table 1. Some information on Jacobi elliptic functions.

Functions	Periods	Zeros	Poles	Residues
$cd\ (z,k)$	$4mK + 2nK'i$	$(2m+1)K + 2nK'i$	$(2m+1)K + (2n+1)K'i$	$(-1)^{m-1}/k$
$cn\ (z,k)$	$4mK + 4nK'i$	$(2m+2n+1)K + 2nK'i$	$2mK + (2n+1)K'i$	$(-1)^{m+n-1}i/k$
$cs\ (z,k)$	$2mK + 4nK'i$	$(2m+1)K + 2nK'i$	$2mK + 2nK'i$	$(-1)^n$
$dc\ (z,k)$	$4mK + 2nK'i$	$(2m+1)K + (2n+1)K'i$	$(2m+1)K + 2nK'i$	$(-1)^{m-1}$
$dn\ (z,k)$	$2mK + 4nK'i$	$(2m+1)K + (2n+1)K'i$	$2mK + (2n+1)K'i$	$(-1)^{n-1}i$
$ds\ (z,k)$	$4mK + 4nK'i$	$(2m+1)K + (2n+1)K'i$	$2mK + 2nK'i$	$(-1)^{m+n}$
$nc\ (z,k)$	$4mK + 4nK'i$	$2mK + (2n+1)K'i$	$(2m+1)K + 2nK'i$	$(-1)^{m+n-1}/k'$
$nd\ (z,k)$	$2mK + 4nK'i$	$2mK + (2n+1)K'i$	$(2m+1)K + (2n+1)K'i$	$(-1)^{n-1}i/k'$
$ns\ (z,k)$	$4mK + 2nK'i$	$2mK + (2n+1)K'i$	$2mK + 2nK'i$	$(-1)^m$
$sc\ (z,k)$	$2mK + 4nK'i$	$2mK + 2nK'i$	$(2m+1)K + 2nK'i$	$(-1)^{n-1}/k'$
$sd\ (z,k)$	$4mK + 4nK'i$	$2mK + 2nK'i$	$(2m+1)K + (2n+1)K'i$	$(-1)^{m+n-1}i/(k.k')$
$sn\ (z,k)$	$4mK + 2nK'i$	$2mK + 2nK'i$	$2mK + (2n+1)K'i$	$(-1)^m/k$

where $0 < k < 1,\ k' = \sqrt{1-k^2}$,

$$F\left(\frac{\pi}{2}, a\right) = \int_0^1 \frac{1}{\sqrt{(1-v^2)(1-a^2v^2)}}\, dv,$$

$K = F\left(\frac{\pi}{2}, k\right)$ and $K' = F\left(\frac{\pi}{2}, k'\right)$.

More information about elliptic functions is provided in [5].
Definition 2.4 [3, p3]. Two matrices $A,\ B \in E^{n\times n}$ (where E denotes the field of elliptic functions of the same periods) are said to be of the same kind (with respect to E) if there exists an invertible matrix $T \in E^{n\times n}$ such that $B = T^{-1}(AT - T')$ and T' is the derivative of matrix T.
Example 2.5. Two matrices $A = \begin{bmatrix} 1 & sn\ t \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 + 2\ sn\ t & 2\ sn\ t \\ -2\ sn\ t & 1 - 2\ sn\ t \end{bmatrix}$ are of the same kind since there exists an invertible matrix $T = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ such that $B = T^{-1}(AT - T')$.

It is obvious that the set E is closed under the operations of addition, subtraction, multiplication, division by non-zero divisor, and differentiation [2, p278]. Now, we consider the set S of all invertible matrices whose entries are elliptic functions of the same periods (or the entries of the matrices are elements of E).

Note 2.6. Since the all the operations of addition, subtraction, multiplication and differentiation on S depend directly on the operations on E, and E is closed under these operations, we can say that S is closed under the all these operations.

Theorem 2.7. Any pair of matrices in S is of the same kind.

Proof. Since the relation "of the same kind" is an equivalence relation [3, p3], then every element of S is of the same kind as itself. Thus we prove the theorem for any distinct pair of matrices in S, and for this purpose we show the elements of S by the set $\{T_i\ ;\ i \in \mathbb{N}\}$. At the first step, we fix $A = T_1$ and choose $T_{m_1} \in \{T_i\ ;\ i \in \mathbb{N},\ i \neq 1\}$ arbitrarily. By **Note 2.6** and the closedness property of S, $T_{m_1}^{-1}$, T'_{m_1} and AT_{m_1} are in S , and then

$$T_{m_1}^{-1}(AT_{m_1} - T'_{m_1}) \in S.$$

We name this element B_{m_1}(so, $B_{m_1} \in S$). Thus for two elements A and B_{m_1} in S there exists $T_{m_1} \in S$ such that

$$B_{m_1} = T_{m_1}^{-1}(AT_{m_1} - T'_{m_1}).$$

Thus A and B_{m_1} are of the same kind. Again, we choose $T_{m_2} \in \{T_i\ ;\ i \in \mathbb{N},\ i \neq 1\ and\ i \neq m_1\}$ and, in the same way as above, we can say that

$$T_{m_2}^{-1}(AT_{m_2} - T'_{m_2}) \in S.$$

We name this element B_{m_2}(so, $B_{m_2} \in S$). Thus for two elements A and B_{m_2} in S there exists $T_{m_2} \in S$ such that

$$B_{m_2} = T_{m_2}^{-1}\left(AT_{m_2} - T'_{m_2}\right).$$

Thus A and B_{m_2} are of the same kind, and so on. In the second step, we let $A = T_2$ and repeat the previous step. We continue by choosing the elements A and T_{m_i}, $(m_i \in \mathbb{N})$, such that A is fixed and T_{m_i} is arbitrary, to complete the proof.

3. Extension of Floquet theorem

Remark 3.1. In [3] it has been mentioned that, in the basic work of Floquet, the independent variable is complex, and the entries of the matrix of the coefficients are analytic functions, and that if these coefficients are not so, then the only possible singularities are isolated singularities. Thus, if we want to extend the Floquet theorem the poles of the entries of the matrix of the coefficients do not affect the extension, because when establishing the theorem, Floquet took it into consideration that some of the functions might have isolated singularities and we extend this theorem depending on the periods of the matrix of the coefficients, and assume that the matrix of the coefficients belongs to S. In other words, in this paper the entries of the matrix of the coefficients are meromorphic and doubly periodic functions.

Now, let $X_1(t), \cdots, X_n(t)$ be n solutions of the linear homogenous system

$$X' = A(t)\, X \tag{1}$$

and $X(t) = \left[[X_1(t)] \cdots [X_n(t)]\right]$, so $X(t)$ is an $n \times n$ matrix solution of (1). If $X_1(t), \cdots, X_n(t)$ are linearly independent, then $X(t)$ is non-singular and is called a fundamental matrix.

Theorem 3.2. Consider the linear homogenous system (1), where $A(t) \in S$. If $W(t)$ is a fundamental matrix of system (1) such that $W(t_0) = I$ (where I represents the identity matrix), then:

i. $W(t + 2mw_1 + 2nw_3)$ are also fundamental matrices of (1), $\forall\, m, n \in \mathbb{Z}$.

ii. Corresponding to every such $W(t)$ there exist an invertible periodic matrix $P(t)$ of period $2mw_1 + 2nw_3$ and a constant matrix R such that $W(t) = P(t)e^{tR}$.

Proof. At the beginning, we mention that our proof will be based on using mathematical double induction. We divide the proof of the theorem into two parts:

i. First we prove the theorem for fundamental periods of $A(t)$.

Case 1: If $m = 1,\ n = 0$, then similar to the proof of the Floquet theorem in [6], $W(t + 2w_1)$ is a fundamental matrix of (1) and there exist an invertible matrix C_0 and a constant matrix R_0 such that

$$C_0 = W(2w_1) = e^{2w_1 R_0},$$

and we define the matrix $P_0(t)$ by

$$P_0(t) = W(t)e^{-tR_0}$$

then it is clear that $P_0(t)$ is periodic of period $2w_1$ and invertible. So

$$W(t) = P_0(t)e^{tR_0}$$

is a fundamental matrix of (1).

Case 2: If $m = 0,\ n = 1$, in the same way $W(t + 2w_3)$ is a fundamental matrix of (1) and there exist C_0^* and R_0^* such that

$$C_0^* = W(2w_3) = e^{2w_3 R_0^*}$$

and we define the matrix $P_0^*(t) = W(t)e^{-tR_0^*}$. Clearly $P_0^*(t)$ is periodic of period $2w_3$ and invertible.

So

$$W(t) = P_0^*(t)e^{tR_0^*}.$$

ii. In this part we prove the theorem for any period of $A(t)$.
Case 1. If $m = n$, suppose $m = n \neq 0$.
Now, for $m = n = 1$,

$$W(t + 2mw_1 + 2nw_3) = W(t + 2w_1 + 2w_3)$$

and since $A(t) \in S$,

$$W'(t + 2w_1 + 2w_3) = A(t)\, W(t + 2w_1 + 2w_3).$$

Then $W(t + 2w_1 + 2w_3)$ is a matrix solution of (1) and, since it is invertible, $W(t + 2w_1 + 2w_3)$ is the fundamental matrix of (1). Therefore, there exists an invertible matrix C_1 such that

$$W(t + 2w_1 + 2w_3) = W(t)\, C_1;$$

By taking $t = t_0 = 0$, $C_1 = W(2w_1 + 2w_3)$. So, there exists a matrix R_1 such that $C_1 = e^{(2w_1+2w_3)R_1}$, see [6, P.139]. We define the matrix $P_1(t) = W(t)e^{-tR_1}$. In this way we easily show that $P_1(t)$ is periodic of period $2w_1 + 2w_3$ and invertible. So

$$W(t) = P_1(t)e^{tR_1}.$$

Suppose that the theorem is true for $= n = k$. This means $W(t + 2kw_1 + 2kw_3)$ is a fundamental matrix of (1) and there exists an invertible matrix $C_k = W(2kw_1 + 2kw_3)$ and a constant matrix R_k such that

$$C_k = e^{(2kw_1+2kw_3)R_k}$$

and

$$W(t) = P_k(t)e^{tR_k}$$

where $P_k(t)$is invertible and periodic of period $2kw_1 + 2kw_3$.
We want to prove that it is true for $m = n = k + 1$. Now,

$$\begin{aligned}
&W'(t + 2(k+1)w_1 + 2(k+1)w_3)\\
&= A(t + 2(k+1)w_1 + 2(k+1)w_3)\,.\,W(t + 2(k+1)w_1 + 2(k+1)w_3)\\
&= A\big((t + 2kw_1 + 2kw_3) + (2w_1 + 2w_3)\big)\,.\,W(t + 2(k+1)w_1 + 2(k+1)w_3)\\
&= A(t + 2kw_1 + 2kw_3)\,.\,W(t + 2(k+1)w_1 + 2(k+1)w_3)\\
&= A(t)\,.\,W(t + 2(k+1)w_1 + 2(k+1)w_3)
\end{aligned}$$

So, $W(t + 2(k+1)w_1 + 2(k+1)w_3)$ is a matrix solution of (1) and also an invertible matrix, then it is the fundamental matrix of (1). Since $W(t)$ and $W(t + 2(k+1)w_1 + 2(k+1)w_3)$ are both fundamental matrices of (1) we must find C_{k+1}, in which

$$W(t + 2(k+1)w_1 + 2(k+1)w_3) = W(t).\,C_{k+1}.$$

For $t = t_0 = 0$,

$$\begin{aligned}
C_{k+1} &= W(2(k+1)w_1 + 2(k+1)w_3)\\
&= W\big(2kw_1 + (2w_1 + 2kw_3 + 2w_3)\big)\\
&= W(2kw_1)W\big(2kw_3 + (2w_1 + 2w_3)\big)\\
&\ \vdots\\
&= W(2kw_1)W(2kw_3)W(2w_1)W(2w_3) = C_k \cdot C_1.
\end{aligned}$$

Hence we have found an invertible matrix C_{k+1}, and for this invertible matrix there exists a matrix R_{k+1} such that

$$C_{k+1} = e^{(2(k+1)w_1+2(k+1)w_3)R_{k+1}}.$$

We define a matrix $P_{k+1}(t)$ by $P_{k+1}(t) = W(t)e^{-tR_{k+1}}$.

$$\begin{aligned}
&P_{k+1}(t + 2(k+1)w_1 + 2(k+1)w_3)\\
&= W(t + 2(k+1)w_1 + 2(k+1)w_3).\,e^{-(t+2(k+1)w_1+2(k+1)w_3)R_{k+1}}\\
&= W(t).\,W(2(k+1)w_1 + 2(k+1)w_3).\,e^{-t\,R_{k+1}}\,.\,e^{-(2(k+1)w_1+2(k+1)w_3)R_{k+1}}\\
&= W(t).\,e^{-t\,R_{k+1}}.
\end{aligned}$$

So, $P_{k+1}(t)$ is periodic of period $2(k+1)w_1 + 2(k+1)w_3$ and it is invertible. Hence

$$W(t) = P_{k+1}(t).\,e^{t\,R_{k+1}};$$

and the theorem is true for all $m, n \in \mathbb{N}$; $m = n$.
Case 2: If $m \neq n$.
a. We fix $m = a$; $a \in \mathbb{N}$; and prove the theorem for $n = 1, 2, 3, \cdots$ by mathematical induction. For $n = 1$, similar to case 1, we can easily show that $W(t + 2aw_1 + 2w_3)$ is a fundamental matrix of (1), and we can find the invertible matrix $C_1^* = W(2aw_1 + 2w_3)$, and for this matrix there exists a matrix R_1^* such that $C_1^* = e^{(2aw_1+2w_3)R_1^*}$. We define a matrix $P_1^*(t)$ by $P_1^*(t) = W(t)e^{-tR_1^*}$. We can also show that it is periodic of period $2aw_1 + 2w_3$, and is an invertible matrix. Then $W(t) = P_1^*(t)e^{tR_1^*}$. Suppose the theorem is true when $n = k$. That means $W(t + 2aw_1 + 2kw_3)$ is the fundamental matrix of (1) and there exist

$$C_k^* = W(2aw_1 + 2kw_3)$$

and R_k^* such that

$$C_k^* = e^{(2aw_1+2kw_3)R_k^*} \text{ and } P_k^*(t) = W(t)e^{-t\,R_k^*}$$

which is invertible and periodic of period $2aw_1 + 2kw_3$.

For $n = k + 1$ we can easily show that $W(t + 2aw_1 + 2(k+1)w_3)$ is a fundamental matrix of (1) and find the invertible matrix

$$C_{k+1}^* = C_k^* . C_0^* = W(2aw_1 + 2kw_3).W(2w_3)$$

and for this invertible matrix there exists a matrix R_{k+1}^* such that

$$C_{k+1}^* = e^{(2aw_1+2(k+1)w_3)R_{k+1}^*}.$$

We define the matrix $P_{k+1}^*(t)$ by $P_{k+1}^*(t) = W(t)e^{-tR_{k+1}^*}$ and show that it is periodic of period $2aw_1 + 2(k+1)w_3$ and is an invertible matrix. Then $W(t) = P_{k+1}^*(t)e^{tR_{k+1}^*}$. Again, we fix $m = a + 1$ and repeat the previous steps

b. We fix $n = b$; $b \in \mathbb{N}$ and prove the theorem for $m = 1, 2, 3, \cdots$ by mathematical induction. Hence the theorem is true for all $m, n \in \mathbb{N}$.

Example 3.3. Consider the linear homogenous system $\begin{cases} x_1' = x_1 + sn\, t\, x_2 \\ x_2' = x_2 \end{cases}$; the fundamental matrix of this system is

$$W(t) = \begin{bmatrix} e^t & \frac{1}{k}\, e^t(-ln(dn\, t + k\, cn\, t) + \ln(1+k)) \\ 0 & e^t \end{bmatrix}.$$

Note that $W(0) = I$, and then, by the above theorem, $W(t + 4K + 8K'i)$ is also the fundamental matrix of the system where

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\, sin^2\theta}}\ ;\ |k| < 1\,,\quad K' = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k'^2\, sin^2\theta}}\ ;\ k' = \sqrt{1-k^2}.$$

Also for $W(t)$ we can find a constant invertible matrix $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and a doubly periodic matrix $P(t)$ of periods $4K + 8K'i$ such that

$$P(t) = \begin{bmatrix} 1 & \frac{-1}{k}\ln\left(\sqrt{1-k^2sn^2t} + k\sqrt{1-sn^2t} + \frac{1}{k}\ln(1+k)\right) \\ 0 & 1 \end{bmatrix}$$

and $W(t) = P(t)e^{tR}$.

Note 3.4. We have only explained the case for the extension of the Floquet theorem when $m, n \in \mathbb{N}$. However it is clear that this extension is true for all $m, n \in \mathbb{Z}$, and we can show this by considering $-2w_1$ and $-2w_3$ as the fundamental periods of $A(t)$. Thus the proof of the extended theorem by depending on this note is completed.

4. Another relative to Floquet theorem

The Halphen theorem is another relative of the Floquet theorem and expresses the fact that, if in a homogeneous linear system of differential equations the matrix of the coefficients are rational functions that are bounded at infinity and if also the general solution is meromorphic, then a fundamental matrix of solutions exists such that its elements are in the form $R(x)\exp(\lambda x)$, in which R is a rational function and λ is a special complex number. Due to the closeness of the Halphen theorem to the Floquet theorem, the rest of the article presents a version of the Halphen theorem as a relative of Floquet theorem.

In this version, the entries of the matrix of the coefficients of system (1) are bounded at a bounded region which is suitably large but contains a finite number of parallelograms.

Definition 4.1. For any elliptic function f on $\mathbb{C}$ with two fundamental periods $2w_1$ and $2w_3$, we define the function f° by

$$f^\circ(t) = f\left(\frac{w}{(4mK + 2nK'i)i}\log t\right);\ w = 2mw_1 + 2nw_3\,,$$

which is a meromorphic function on $\mathbb{C} - \{0\}$.

Remark 4.2. Since the entries of $A(t)$ are elliptic functions of two periods $2w_1$ and $2w_3$, then the z −plane will be divided into an infinite number of parallelograms and period strips by these two periods, in such a way that each two non-parallel period strips will intersect each other in one parallelogram. Let L_1 and L_2 be two period strips which intersect each other in the period parallelogram denoted by Δ.

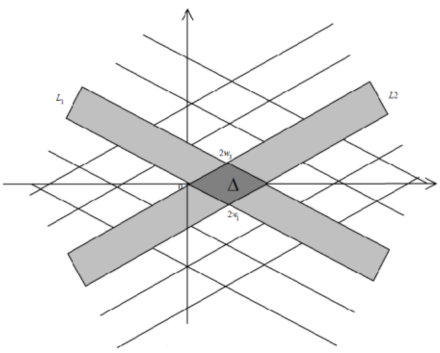

Figure 1. Period parallelogram generated by the intersection of two non-parallel period strips.

Definition 4.3. We define a bounded region L by taking the period strip L_2 which is suitably large but contains a finite number of parallelograms.

Theorem 4.4. Suppose that all entries of the matrix of the coefficients $A(t)$, of system (1) are bounded at L. If system (1) has only meromorphic solutions, then there exists a constant $(n \times n)$ matrix J in Jordan normal form and an $(n \times n)$ matrix R° whose entries are rational functions over $\mathbb{C}$, such that the following statements hold:

i. Suppose that there are non-negative integers $v_1, \cdots, v_{r-1}$ such that $\lambda,\ \lambda + iv_1, \cdots, \lambda + iv_{r-1}$ are all the eigenvalues of $A^\circ(0)$ which are equal to λ modulo . Then λ is an eigenvalue of J with algebraic multiplicity r.
ii. System (1) has a fundamental matrix given by

$$X(t) = \left[R^\circ(\exp(i(4mK + 2nK'i)t/w)) \cdot exp\big((4mk + 2nk'i)Jt/w\big)\right]. \tag{2}$$

Conversely, suppose that R° is an invertible $(n \times n)$ matrix whose entries are meromorphic functions and J is a constant $(n \times n)$ matrix. Then $X(t)$ as in the equation (2) is a fundamental matrix of system (1) where $A(t) \in S$ and is of the same kind as a matrix whose entries are bounded at L.

Proof. We define the function f° as in definition 4.1. On the other hand, in [1] it was mentioned that if f is a doubly periodic function, then f does not have finitely many poles in the period strip, and hence does not have definite limits at the ends of the period strip, and consequently we cannot say f° is a rational function. So, to deal with this, we define the bounded region L as in definition 4.3. Now, the theorem can be proved by taking $= 4mK + 2nK'i$, and the rest of the proof is similar to the proof of the theorem 1 in [3]. To avoid our repeating the technical steps of the proof and for better understanding, it is necessary that the reader to refer to [3].

Example 4.5. This example explains the converse of the above theorem.
Consider the linear homogenous system

$$\begin{aligned} x_1' &= x_1 + \sqrt{1 - sn^2\big(f(t)\big)} \cdot \sqrt{1 - k^2 sn^2\big(f(t)\big)} \cdot x_2, \\ x_2' &= x_2 \end{aligned} \tag{3}$$

and let $R^0 = \begin{bmatrix} 1 & sn\,(f(t)) \\ 0 & 1 \end{bmatrix}$ (where $f(t) = \frac{1}{i}\log e^{it}$) be an invertible matrix whose entries are meromorphic functions and $J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be a constant matrix, then by the above theorem

$$R^0 \times e^{tJ} = \begin{bmatrix} 1 & sn\,(f(t)) \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^t & e^t sn\,(f(t)) \\ 0 & e^t \end{bmatrix}$$

is the fundamental matrix of system (3) and it is clear that the matrix of the coefficients of the system

$$A(t) = \begin{bmatrix} 1 & \sqrt{1 - sn^2\big(f(t)\big)} \cdot \sqrt{1 - k^2 sn^2\big(f(t)\big)} \\ 0 & 1 \end{bmatrix}$$

is of the same kind as matrix B that is bounded at L and $B =$

$$\begin{bmatrix} 3 \cdot \sqrt{1 - sn^2\big(f(t)\big)}\ \sqrt{1 - k^2 sn^2\big(f(t)\big)} + 1 & -\frac{3}{2} + \frac{9}{2}\sqrt{1 - sn^2\big(f(t)\big)}\ \sqrt{1 - k^2 sn^2\big(f(t)\big)} + 1 + \frac{3}{2} \\ -2 \cdot \sqrt{1 - sn^2\big(f(t)\big)}\ \sqrt{1 - k^2 sn^2\big(f(t)\big)} & 1 - 3 \cdot \sqrt{1 - sn^2\big(f(t)\big)}\ \sqrt{1 - k^2 sn^2\big(f(t)\big)} \end{bmatrix}.$$

5. Conclusion

In this work, doubly periodic functions are introduced generally and some Jacobi elliptic functions are specifically illustrated. The concepts of matrices of the same kind and additionally doubly periodic functions were applied for the extension of Floquet theorem. Furthermore, there is a detailed description of any pair of matrices, the entries of which are elliptic functions of the same periods, which are of the same kind. In addition it has been proved that if $W(t)$ is a fundamental matrix of system (1), then there exists an invertible doubly periodic matrix $P(t)$ and a constant matrix R such that $W(t) = P(t)e^{tR}$. Finally, another theorem that is related to the theorem of Floquet is presented, with an example to explain it.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgment

The authors thank the anonymous reviewers for their valuable suggestions, helpful comments, and constructive criticisms for improving the manuscript during the process of preparing this article for publication.

References

1. Mohammad, N.H. Extension of Floquet theorem based on elliptic functions. M.Sc. Thesis, Salahaddin University, Hawler, Kurdistan Region-Iraq, 2003.
2. Gesztesy, F. and Weikard, R. Elliptic algebro-geometric solutions of the KDV and AKNS hierarchies-an analytic approach. *Bulletin of the American Mathematical Society*, 1998, **35(4)**, 271-317.
3. Weikard, R. Floquet theory for linear differential equations with meromorphic solutions. *Electronic Journal of Qualitative Theory of Differential Equations.*, 2000, **8**, 1-6.
4. Chouikha, A.R. On properties of Jacobi elliptic functions and applications. *Journal of Nonlinear Mathnetical Physics*, 2005, **12(2)**, 162-169.
5. Freitag**,** E. and Busam**,** R. Complex Analysis (2nd Edition), Springer-Verlag, Berlin, 2005.
6. Myint-u, T. Ordinary Differential Equations. Elsevier North Holland, New York, 1978.

Received 1st January 2020
Accepted 26 July 2020