SQU Journal for Science, 2015, **20(2)**, 19-30

Numerical Wave Solutions for Nonlinear Coupled Equations using Sinc-Collocation Method

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ABSTRACT: In this paper, numerical solutions for nonlinear coupled Korteweg-de Vries(abbreviated as KdV) equations are calculated by the Sinc-collocation method. This approach is based on a global collocation method using Sinc basis functions. The first step is to discretize time derivative of the KdV equations by a classic finite difference formula, while the space derivatives are approximated by a θ- weighted scheme. Sinc functions are used to solve these two equations. Soliton solutions are constructed to show the nature of the solution. The numerical results are shown to demonstrate the efficiency of the newly proposed method.

Keywords: Nonlinear coupled KdV equations; Sinc-collocation method; Soliton solutions.

حلول الموجة العددية لمعادلات الدوال غير الخطية باستخدام طريقة سينك-التجميعية

كامل الخالد

ملخص: تم في هذا البحث إيجاد الحلول العددية لمعادلات الدوال غير الخطية والمسماة كورتيج دو فريس (مختصر KdV) بطريقة سينك-التجميعية. يستند هذا النهج على طريقة تجميعية شاملة باستخدام دوال السينك الأساسية. تم في الخطوة الأولى تقطيع مشتقة الزمن للمعادلات KdV وفقا لصيغة فرق المحدود الكلاسيكية، بينما تم تقريب المشتقات الفضائية بطريقة نظام الموازنة. وتم حل هاتين المعادلتين باستخدام دوال سينك. كما تم إيجاد وحساب حلول الموجة المنعزلة لإظهار طبيعة الحل. وأظهرت النتائج العددية كفاءة الطريقة المقترحة.

كلمات مفتاحية: معادلات KdV المقرونة غير الخطية، حلول الموجة المنعزلة، طريقة سينك-التجميعية.

1. Introduction

Nonlinear partial differential equations appear in many branches of physics, engineering and applied mathematics. The coupled KdV equations, recently, have been an attractive research area for scientists, because of their many applications in scientific fields and many studies have been reported in the literature, see for example [1-5]. In this paper, we study the coupled KdV equations that were introduced by Hirota-Satsuma [6]

$$u_t = -\alpha u_{xxx} - 6\alpha u u_x + 2\mu v v_x,\ v_t = -\beta v_{xxx} - 3\beta u v_x, \tag{1}$$

subject to the initial conditions

$$u(x,0) = f(x),\ \ v(x,0) = g(x),\ a \le x \le b \tag{2}$$

and boundary conditions

$$u(a,t) = f_1(t),\ u(b,t) = f_2(t),\ \ v(a,t) = g_1(t),\ v(b,t) = g_2(t), \tag{3}$$

where $u(x,t)$ and $v(x,t)$ are real-valued scalar functions, t is time, and x is a spatial variable. The equations (1)-(3) describe interactions of two long waves with different dispersion relations. Many powerful methods have been developed to find solutions (exact, or numerical) of such nonlinear evolution equations. These include the Adomian decomposition method [7] and the collocation method [8]. The soliton solutions for this system are constructed by Fan [9].

Mesh-free methods are the topic of recent research in many areas of computational science and approximation theory. Over the past several years mesh-free approximation methods have found their way into many different application areas ranging from engineering to the numerical solution of differential equations. A meshless method does not require a grid, and only makes use of a set of scattered collocation points. In [10,11], the authors propose a mesh-free collocation method and formulate simple classical radial basis functions for the numerical solution of the KdV equation, and coupled KdV equations. In [12], the authors employ a mesh-free technique for the computational solution of the two-dimensional coupled Burgers' equations. They combine the collocation method using the radial basis functions with first-order accurate forward difference approximation to obtain a mesh-free solution of the coupled Burgers' equations. A mesh-free technique was presented in [13] for the solution of the generalized regularized long wave equation, the collocation being founded using Sinc function basis. The purpose of this paper, and following [11], is to elaborate a special collocation method based on Sinc function basis for spatial derivatives, and classic finite difference formulae for time derivatives, to obtain numerically the traveling wave solutions of the KdV system in (1)-(3). The Sinc-collocation method will be used in the space direction. The main idea is to replace derivatives by their Sinc approximations. The ease of implementation coupled with the exponential convergence rate have demonstrated the viability of this method. Sinc functions are discussed in Stenger [14] and by Lund and Bowers [15].

The layout of the paper is as follows. In section2, we briefly review some general concepts of the Sinc function that are necessary for the formulation of the discrete system. In section 3, we discuss the mesh-free method together with the Sinc-collocation discretization of the coupled KdV equations. Section 4 is devoted to the stability of the method, by using a linearized stability analysis. Finally, numerical experiments are presented and some comparisons are made in section 5. Some concluding remarks are given in the final section 6.

2. Sinc-Collocation

The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [14,10]. The Sinc function is defined on the whole real line R by

$$\text{sinc}\,(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{4}$$

Recall that a radial basis function is a function whose value depends only on the distance of its input to a central point. For a series of nodes equally spaced h apart, the Sinc function can be written as a radial basis function:

$$S_j(x) = \text{sinc}\left(\frac{x - jh}{h}\right), \quad j = 0, \mp 1, \mp 2, ... \tag{5}$$

The Whittaker cardinal function $C(f\,,h)$ of a function f is defined as

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh) S_j(x).$$

Whenever this series converges, f is approximated by using the finite number of terms. So, for positive integer N, define

$$C_N(f,h)(x) = \sum_{j=-N}^{N} f(jh) S_j(x). \tag{6}$$

Definition 2.1 *Let $d > 0$, and let D_d denote the region $\{z = x + iy, |\, y\,| < d\}$ in the complex plane C, and ϕ the conformal map of a simply connected domain D in the complex plane domain onto D_d such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where a and b are the boundary points of D. Let ψ denote the inverse map of ϕ, and let the arc Γ, with end points a and b $(a, b \notin \Gamma)$, be given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ be given by $x_k = \psi(kh)$, $z \in Z$ and $\rho(z) = \exp(\phi(z))$.*

Hence, the numerical process developed in the domain containing the whole real line can be carried over to infinite interval by the inverse map. The approximation of the n^{th} derivatives of $f\,(x)$ by the Sinc expansion is given by

$$f^{(n)} \approx \sum_{j=-N}^{N} f(jh) \frac{d^n}{dx^n}[S_j(x)]. \tag{7}$$

The derivatives of Sinc functions evaluated at the nodes will be needed [14,15]. In particular, the following convenient notation will be useful in formulating the discrete system.

$$\delta_{jk}^{(0)} = [S_j(x)]\,|_{x = x_k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{8}$$

$$\delta_{k-j}^{(1)} = \frac{d}{dx}[S_j(x)]|_{x=x_k} = \begin{cases} 0, & j=k \\ \dfrac{(-1)^{jk}}{h(k-j)}, & j \neq k \end{cases} \tag{9}$$

$$\delta_{k-j}^{(3)} = \frac{d^3}{dx^3}[S_j(x)]|_{x=x_k} = \begin{cases} 0, & j=k \\ \dfrac{(-1)^{jk}[6-(k-j)^2\pi^2]}{(k-j)^3}, & j \neq k \end{cases} \tag{10}$$

where the collocation points are $x = x_k$. In practice, we need to use a finite number of terms in the series (7), say $j = -N, ..., N,$ where N is the number of Sinc grid points. For a restricted class of functions known as the Paly-Weiner class, which are entire functions, the Sinc interpolation and quadratic formulas are exact [14]. A less restrictive class of functions which are analytic only on an infinite strip containing the real line and which allow specific growth restrictions have exponentially decaying absolute errors in the sinc approximation. In order to state the convergence theorem of the Sinc-collocation method, we introduce the following notation and definition.

Definition 2.2 *For all* $0 < \varepsilon < 1$, *let* $D_d(\varepsilon)$ *be defined by*

$$D_d(\varepsilon) = \{z \in C : |Rez| < 1/\varepsilon, \ |Imz| < d(1-\varepsilon)\}. \tag{11}$$

Let $B(D_d)$ *be the Hardy space over the region* D_d *, i.e., the set of all functions such that*

$$\lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(z)||dz| < \infty. \tag{12}$$

The properties of functions in $B(D_d)$ and detailed discussions are given in [14]. We recall the following theorem for our convergence purposes.

Theorem 2.1[14] *Let* α, β *and* d *be positive constants. Assume that*

1. $f \in B(D_d)$
2. f *decays exponentially on the real line, that is,* $|f(x)| \le \alpha \exp(-\beta|x|), x \in R.$

Then, we have

$$\sup\left| f^{(n)}(x) - \sum_{j=-N}^{N} f_j S_j^{(n)}(x) \right| \le C_1 N^{(n+1)/2} \exp(-\sqrt{\pi d \beta N})$$

for some constant C_1*, where the mesh size* h *is taken as* $h = \sqrt{\pi d/(\beta N)}.$

The above theorem states that if f is an analytic function on an infinite strip containing the real line, and satisfies some kind of decaying conditions, then the function f together with it derivatives can be approximated by Sinc function methodology with error of exponential order. Therefore, in order to approximate the solution of the KdV system (1) using Sinc basis, we should start with the assumption that the initial conditions in (3) belong to class $B(D_d)$. The matrices $I^{(0)}, I^{(1)}, I^{(3)}$ will appear in the final discrete system, and in order to study the stability of the Sinc-collocation method, we should find some bound for the eigenvalues of these matrices. The matrix $I^{(0)}$ is just the identity matrix which has eigenvalue 1. For eigenvalue bounds for the Toeplitz matrices $I^{(1)}, I^{(3)},$ we state the following theorem.

Theorem 2.2[14]

1. *The* $m \times m$ *matrix* $I^{(1)}$ *is a singular skew-symmetric matrix, if its eigenvalues are denoted by* $\{i\lambda_j^{(1)}\}_{j=-N}^{N}$ *, then* $-\pi \le \lambda_{-N}^{(1)} \le ... \le \lambda_N^{(1)} \le \pi.$
2. *The* $m \times m$ *matrix* $I^{(3)}$ *is a singular skew-symmetric matrix, if its eigenvalues are denoted by* $\{i\lambda_j^{(3)}\}_{j=-N}^{N}$ *, then* $-\pi^3 \le \lambda_{-N}^{(3)} \le ... \le \lambda_N^{(3)} \le \pi^3.$

3. Construction of the Method

Consider the nonlinear coupled KdV equations in (1), subject to the initial and boundary conditions (2)-(3). To implement the Sinc-collocation method, we discretize the time derivative of the nonlinear coupled KdV equations using the classic finite difference formula, and space derivatives by the θ- weighted $(0 \le \theta \le 1)$ scheme successive two time levels n and $n+1$

$$\frac{u^{n+1}-u^n}{\delta t} = -\alpha\theta(u_{xxx}{}^{n+1} - \alpha(1-\theta)(u_{xxx})^n - 6\alpha\theta(uu_x)^{n+1} - 6\alpha(1-\theta)(uu_x)^n + 2\mu(vv_x)^{n+1} \tag{13}$$

and,

$$\frac{v^{n+1}-v^n}{\delta t} = -\beta\theta(v_{xxx})^{n+1} - \beta(1-\theta)(v_{xxx})^n - 3\beta(uv_x)^n \tag{14}$$

where $u^n = u(x,t^n)$ is the value of the solution at the n^{th} time step, and $t^n = t^{n-1} + \delta t,$ where δt is a time step size. The nonlinear term $(uu_x)^{n+1}$ must be linearized before continuing. This can be accomplished by using the following formula which is obtained by applying the Taylor expansion, as follows

$$(u_x)^{n+1} \approx (u_x)^n + \delta t \frac{u_x^{n+1} - u_x^n}{\delta t} + O(\delta t^2)\cdot$$

Thus,

$$(uu_x)^{n+1} \approx (uu_x)^n + \delta t[(u_t)^n u_x^n + (u)^n u_{xt}^n] + O(\delta t^2) \tag{15}$$

which can be simplified to

$$(uu_x)^{n+1} \approx (uu_x)^n + \delta t[u_x^n \frac{(u)^{n+1}-(u)^n}{\delta t} + u^n \frac{u_x^{n+1}-u_x^n}{\delta t}] + O(\delta t^2)\cdot \tag{16}$$

Finally, we arrive at the linearization

$$(uu_x)^{n+1} \approx (u)^{n+1}u_x^n + u_x^{n+1}u^n - u^n u_x^n \tag{17}$$

so that equations (14)-(15) can be rewritten as

$$\begin{aligned} &u^{n+1} + \alpha\delta t\theta(u_{xxx})^{n+1} + 6\alpha\delta t\theta(u^n u_x^{n+1} + u^{n+1}u_x^n) \\ &= u^n - \alpha\delta t(1-\theta)u_{xxx}^n - 6\alpha\delta t(1-\theta)u^n u_x^n + 6\alpha\delta t\theta u^n u_x^n + 2\mu\delta t v^n v_x^n \end{aligned} \tag{18}$$

and

$$v^{n+1} + \beta\delta t\theta(v_{xxx})^{n+1} = v^n - \beta\delta t(1-\theta)(v_{xxx})^n - 3\beta\delta t u^n v_x^n \tag{19}$$

where u^n and v^n are the n^{th} iterates of the approximate solutions. Now the space variable is discretized upon the use of Sinc-collocation at the points

$$\{x_1 = a, ..., x_i + a + (i-1)h, ..., x_N = b\},\ h = \frac{|b-a|}{N-1}. \tag{20}$$

The solution of equation (13) is interpolated and approximated by means of the Sinc functions as

$$u^n(x) = \sum_{j=1}^{N} u_j^n S_j(x), \quad v^n(x) = \sum_{j=1}^{N} v_j^n S_j(x) \tag{21}$$

where

$$S_j(x) = \operatorname{sinc}(\frac{x-(j-1)h-a}{h}). \tag{22}$$

The unknown parameters u_j, v_j in equation (21) are to be determined by the collocation method. Therefore, for each collocation point x_i in (20), equation (22) can be written as

$$u^n(x_i) = \sum_{j=1}^{N} u_j^n S_j(x_i),\ v^n(x_i) = \sum_{j=1}^{N} v_j^n S_j(x_i),\ i = 1, ..., N. \tag{23}$$

Substituting equation (23) into equations (18) and (19), we get

$$\begin{aligned}&\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)+\alpha\delta t\theta\sum_{j=1}^{N}u_j^{n+1}S_j''(x_i)+6\alpha\delta t\theta[\sum_{j=1}^{N}u_j^{n}S_j(x_i)\sum_{j=1}^{N}u_j^{n+1}S_j'(x_i)\\&+\sum_{j=1}^{N}u_j^{n}S_j'(x_i)\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)]=\sum_{j=1}^{N}u_j^{n}S_j(x_i)-\alpha\delta t(1-\theta)\sum_{j=1}^{N}u_j^{n}S_j''(x_i)\\&-6\alpha\delta t(1-\theta)\sum_{j=1}^{N}u_j^{n}S_j(x_i)\sum_{j=1}^{N}u_j^{n}S_j'(x_i)+6\alpha\delta t\theta\sum_{j=1}^{N}u_j^{n}S_j(x_i)\sum_{j=1}^{N}u_j^{n}S_j'(x_i)\\&+2\mu\delta t\sum_{j=1}^{N}v_j^{n}S_j(x_i)\sum_{j=1}^{N}v_j^{n}S_j'(x_i).\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\sum_{j=1}^{N}v_j^{n+1}S_j(x_i)+\beta\delta t\theta\sum_{j=1}^{N}v_j^{n+1}S_j''(x_i)=\sum_{j=1}^{N}v_j^{n}S_j(x_i)-\\&\beta\delta t(1-\theta)\sum_{j=1}^{N}v_j^{n}S_j''(x_i)-3\beta\delta t\sum_{j=1}^{N}u_j^{n}S_j(x_i)\sum_{j=1}^{N}v_j^{n}S_j'(x_i)\cdot\end{aligned}\tag{25}$$

Equations (24) and (25) are used for all interior points $x=x_i,\ i=2,\dots,N-1$, where primes in these equations denote differentiation with respect to the variable x. The boundary condition given by equation (17) for the boundary points $x=x_i, i=1,N$ can be written as

$$\sum_{j=1}^{N}u_j^{n+1}S_j(x_i)=f_1(t^{n+1}),\quad \sum_{j=1}^{N}u_j^{n+1}S_j(x_i)=f_2(t^{n+1})\tag{26}$$

and

$$\sum_{j=1}^{N}v_j^{n+1}S_j(x_i)=g_1(t^{n+1}),\quad \sum_{j=1}^{N}v_j^{n+1}S_j(x_i)=g_2(t^{n+1}).\tag{27}$$

To obtain matrix representation of the expression in equation (24) and (25), we introduce the following matrix and vector notations

$$\begin{aligned}U^n&=[u_1^n,u_2^n,\dots,n_N^n]^T,\\I^{(0)}&=S_j(x_i),\ \ i,j=1,\dots,N,\\I^{(1)}&=S_j'(x_i),\ \ i,j=1,\dots,N,\\I^{(3)}&=S_j''(x_i),\ \ i,j=1,\dots,N.\end{aligned}\tag{28}$$

Note that the matrices $I^{(1)},I^{(3)}$ are skew symmetric. The system of equations (24)-(27) can be solved for unknown parameters u_j,v_j in equation (21) simultaneously, and then the solutions for u and v can be obtained from equation (21). Equations (24)-(27) can be written in matrix form as

$$\begin{aligned}&[I^{(0)}+\alpha\delta t\theta I^{(3)}+6\alpha\delta t\theta[U^n*I^{(1)}+U_x^n*I^{(0)}]]u^{n+1}\\&=[I^{(0)}-\alpha(1-\theta)\delta tI^{(3)}-6\alpha(1-\theta)\delta tU^n*I^{(1)}+6\alpha\delta t\theta U^n*I^{(1)}+2\mu\delta tV^n*I^{(1)}]u^n+F^{n+1}\end{aligned}\tag{29}$$

and

$$[I^{(0)}+\beta\delta t\theta I^{(3)}]v^{n+1}=[I^{(0)}-\beta(1-\theta)\delta tI^{(3)}-3\beta\delta tU^n*I^{(1)}]v^n+G^{n+1}\tag{30}$$

where $*$ stands for component by component multiplication, and $F^{n+1}=[f_1(t^{n+1}),0,\dots,0,f_2(t^{n+1})]^T$, and $G^{n+1}=[g_1(t^{n+1}),0,\dots,0,g_2(t^{n+1})]^T$. Equations (29) and (30) can be written in a more compact form as

$$A_1u^{n+1}=B_1u^n+F^{n+1}\tag{31}$$

and

$$A_2v^{n+1}=B_2v^n+G^{n+1},\tag{32}$$

where

$$A_1 = I^{(0)} + \alpha\delta t\theta I^{(3)} + 6\alpha\delta t\theta[U^n * I^{(1)} + U_x^n * I^{(0)}],$$
$$B_1 = I^{(0)} - \alpha(1-\theta)\delta t I^{(3)} - 6\alpha(1-\theta)\delta t U^n * I^{(1)} + 6\alpha\delta t\theta U^n * I^{(1)} + 2\mu\delta t V^n * I^{(1)},$$
$$A_2 = I^{(0)} + \beta\delta t\theta I^{(3)},$$
$$B_2 = I^{(0)} - \beta(1-\theta)\delta t I^{(3)} - 3\beta\delta t U^n * I^{(1)}.$$

Equation (21) can written in matrix form as

$$U^n = I^{(0)}u^n,\ V^n = I^{(0)}v^n. \tag{33}$$

From equation (31), we get

$$u^{n+1} = A_1^{-1}B_1u^n + A_1^{-1}F^{n+1}.$$

Combining the above equation together with equation (33), we arrive at

$$U^{n+1} = I^{(0)}A_1^{-1}B_1u^n + I^{(0)}A_1^{-1}F^{n+1}. \tag{34}$$

Similarly, using equations (32) and (33) we arrive at

$$V^{n+1} = I^{(0)}A_2^{-1}B_2v^n + I^{(0)}A_2^{-1}G^{n+1}. \tag{35}$$

We can obtain the coefficients of the approximate solution by solving the system in equations (34)-(35) using any iterative technique. For the convergence of the method, we state the following two theorems.

Theorem 3.1 *Let the function $u(x,t)$ be as in equation (1) with the initial condition as in equation (2), and let the matrix U be defined as in (34). Then for a sufficiently large N, there exists a constant C independent of N such that*

$$\sup_{(x_j,t^n)} \| u(x_j,t^n) - U \| \le CN^2 \exp(-\sqrt{\pi d\beta N}).$$

Theorem 3.2 *Given a constant $R > 0$, there is a constant $T > 0$ such that if $\|U^1 - U^0\| \le R/2$, then the iterative scheme (34) converges to the unique solution.*

The proof of the above two theorems are immediate from [16,17]. We would like to mention here that for Theorem 3.1, we use the second part of Theorem 2.1 for $n=1$ and $n=3$, which is a simple modification of Theorem 3.2 in [2]. For Theorem 3.2, we use contraction mapping of the iteration scheme given in equation (34) and apply fixed point theorem to prove convergence. Interested readers may follow [16] for a detailed analysis

4. Stability Analysis

In this section, we present an analysis of the stability of the Sinc-collocation method for solving the coupled KdV system using spectral radius matrices. Following the method outlined in [11], let U, V be the exact, and $\tilde{U}, \tilde{V}$ be the numerical solutions of the coupled system in (1). Define the error vectors $\varepsilon_u^n = U - \tilde{U}$, and $\varepsilon_v^n = V - \tilde{V}$. Using equations (34) and (35) the errors ε_u^n, ε_v^n can be written as

$$\varepsilon_u^{n+1} = U^{n+1} - \tilde{U}^{n+1} = I^{(0)}A_1^{-1}B_1\varepsilon_u^n,\ \varepsilon_v^{n+1} = V^{n+1} - \tilde{V}^{n+1} = I^{(0)}A_2^{-1}B_2\varepsilon_v^n. \tag{36}$$

For stability of the method, we need $\varepsilon_u^n \to 0$, and $\varepsilon_v^n \to 0$ for large values of n. Therefore, the scheme is considered numerically stable if $\rho(I^{(0)}A_1^{-1}B_1) < 1$, and $\rho(I^{(0)}A_2^{-1}B_2) < 1$, where $\rho(.)$ denotes the spectral radius. The Sinc matrices $I^{(1)}, I^{(3)}$ are contained in the matrices A_1, A_2, B_1 and B_2, where the bounds for the eigenvalues of these matrices are given in Theorem 2.2.

Therefore, stability is assured if

$$\left| \frac{1 + \alpha\delta t\theta\lambda_3 + 6\alpha\delta t\theta[\lambda_{1N} + \lambda_{2N}]}{1 - \alpha(1-\theta)\delta t\lambda_3 - 6\alpha(1-\theta)\delta t\lambda_{1N} + 6\alpha\delta t\theta\lambda_{1N} + 2\mu\delta t\lambda_{1N}} \right| < 1 \tag{37}$$

and

$$\left|\frac{1+\beta\delta t\theta\lambda_3}{1-\beta(1-\theta)\delta t\lambda_3-3\beta\delta t\lambda_{1N}}\right|<1 \tag{38}$$

where we have used the numbers $1, \lambda_1, \lambda_3, \lambda_{1N}, \lambda_{2N}$ being eigenvalues of the matrices $I^{(0)}, I^{(1)}, I^{(3)}, U^n * I^{(1)}, U^n * I^{(0)}$ respectively. Using the upper bound for the eigenvalues in Theorem 2.2, together with the fact that $\lambda_1 = i\,|\lambda_1|$, $\lambda_3 = i\,|\lambda_3|$, and if $\lambda_{1N}, \lambda_{2N}$ are complex, then after algebraic manipulation (see, [13,10,11]), the conditions (37) and (38) must hold for all eigenvalues of the respective matrices. Having $1/2 \le \theta < 1$ is necessary, but not sufficient, to guarantee the stability of the Sinc collocation method.

5. Numerical Results

Choosing examples with known solutions allows for a more complete error analysis. In order to assess the advantages of the proposed method, in terms of accuracy and efficiency for solving nonlinear coupled KdV equations, the following examples are presented in this section.

Example 5.1 *In this example, we have to apply our scheme to solve equation (1) with* $\beta = 1, \alpha \ne 1/2$, *and* $\alpha\mu > 0$, *with the initial conditions*

$$u(x,0) = -\frac{1+\alpha}{3+6\alpha}k^2 + 4k^2\frac{e^{kx}}{(1+e^{kx})^2},\ \ v(x,0) = \frac{Me^{kx}}{(1+e^{kx})^2} \tag{39}$$

Where $M = \sqrt{\frac{24\alpha}{\mu}}k^2$, *and* k *is an arbitrary constant. The exact solution is given by [7]*

$$u(x,t) = -\frac{1+\alpha}{3+6\alpha}k^2 + 4k^2\frac{e^{k(x+ct)}}{(1+e^{k(x+ct)})^2},\ \ v(x,t) = \frac{Me^{k(x+ct)}}{(1+e^{k(x+ct)})^2}. \tag{40}$$

The computations associated with the example were performed using Mathematica. In our computational work, we take $\alpha = 1.5, \mu = 0.1, c = 0.1, k = 0.1$, and two different time step sizes $\delta t = 0.1$ and $\delta t = 0.05$ through the interval $[-30, 30]$ and $N = 160$ for the set of collocation points as in equation (20). The accuracy of the scheme is measured by using the following two error norms [1, 11]

$$L_2 = \| u - \tilde{u} \|_2 = \sqrt{h\sum_{j=1}^{N}| u_j - \tilde{u}_j |^2}, \quad L_\infty = \| u - \tilde{u} \|_\infty = \max_{1\le j\le N} | u_j - \tilde{u}_j |$$

where u and $\tilde{u}$ represent the exact and approximate solutions, respectively, and h is the minimum distance between any two points in equation (21), similarly for the v solution. The pointwise rate of convergence in time is also calculated by using the following formula [1,11]

$$\text{Order} = \frac{\log_{10}(\| u_{exact} - u_{\delta t_j} \| / \| u_{exact} - u_{\delta t_{j+1}} \|)}{\log_{10}(\delta t_j / \delta t_{j+1})}$$

where u_{exact} is the exact solution, and $u_{\delta t_j}$ is the numerical solution with time step size δt_j. In Table 1, the number of collocation points is $N = 160$, and the time step size $\delta t = 0.8, 0.4, 0.2, 0.1, 0.05, 0.025, 0.01$ is varied to compute the time rate of convergence when $\theta = 1/2$. It can be noted from Table 1 that the method has order of convergence 2. The accuracy of the proposed method is demonstrated for the absolute errors for solution of (1) with their exact solutions. Table 2 reports the supremum norm error between the exact solution (40) and our approximate solution compared with the results in [11]. A clear conclusion can be drawn from the numerical results in Figures 1, 2 and Table 2 that Sinc methodology provides highly accurate numerical solutions.

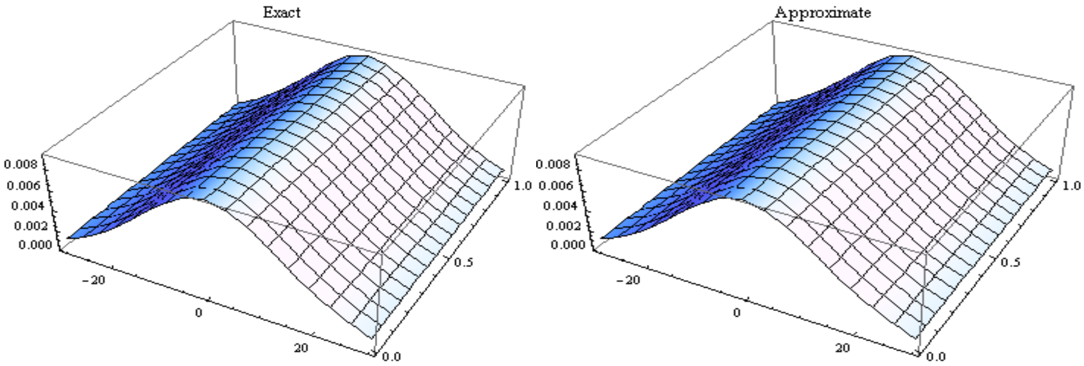

Figure 1. The exact and approximate solutions for $u(x,t)$ in Example 5.1.

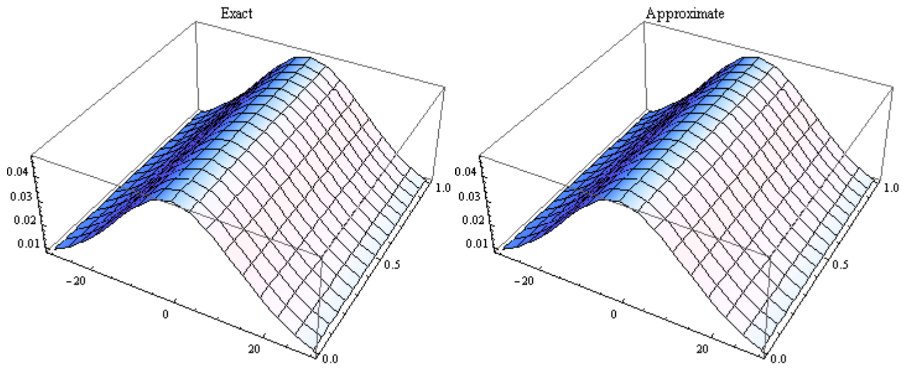

Figure 2. The exact and approximate solutions for $v(x,t)$ in Example 5.1.

Table 1. Time rate of convergence solution of equation (1): $t=16, \theta=1/2, N=160, -30\le x\le 30\cdot$

δt	L_∞	Order	L_2	Order
0.8	3.10524×10^{-3}	–	1.10518×10^{-2}	–
0.4	1.22201×10^{-3}	1.96166	3.22148×10^{-3}	1.96453
0.2	5.35209×10^{-4}	1.99086	8.35020×10^{-4}	1.99486
0.1	1.49743×10^{-4}	1.99748	4.49278×10^{-4}	1.99782
0.05	8.66014×10^{-5}	1.99835	1.65082×10^{-4}	1.99892
0.025	6.84252×10^{-5}	1.99809	5.82611×10^{-5}	1.99212
0.010	2.04701×10^{-5}	1.98971	1.02058×10^{-5}	1.98775

Table 2. Numerical results for Example 5.1 when $\delta t = 0.001$.

x	$\lvert u-\tilde{u}\rvert$	$\lvert u-\tilde{u}\rvert$ as in [13]	$\lvert v-\tilde{v}\rvert$	$\lvert v-\tilde{v}\rvert$ as in [13]
-4	2.039×10^{-8}	1.766×10^{-7}	9.672×10^{-8}	1.239×10^{-8}
-3	1.565×10^{-8}	1.368×10^{-6}	7.423×10^{-8}	9.832×10^{-8}
-2	1.060×10^{-8}	1.056×10^{-5}	5.031×10^{-8}	8.829×10^{-7}
-1	5.356×10^{-9}	2.966×10^{-5}	2.540×10^{-8}	1.023×10^{-5}
0	2.485×10^{-13}	1.367×10^{-4}	1.179×10^{-12}	2.001×10^{-6}
1	5.353×10^{-9}	3.897×10^{-5}	2.540×10^{-8}	1.195×10^{-5}
2	1.060×10^{-8}	1.241×10^{-5}	5.031×10^{-8}	1.252×10^{-6}
3	1.565×10^{-8}	1.666×10^{-5}	7.423×10^{-8}	1.182×10^{-6}
4	2.039×10^{-8}	7.380×10^{-6}	9.672×10^{-8}	5.219×10^{-7}

Example 5.2 *In this example, we solve the problem*

$$u_t = -u_{xxx} - uu_x - 2vv_x,\ v_t = -v_{xxx} - uv_x \tag{41}$$

subject to the initial conditions

$$u(x,0) = \frac{3}{4}sech^2[\frac{1}{2}\sqrt{\frac{1}{2}}x],\ \ v(x,0) = \frac{3}{4}\sqrt{\frac{1}{2}}\tanh^2[\frac{1}{2}\sqrt{\frac{1}{2}}x],\ a \le x \le b \tag{42}$$

and boundary conditions $u(a,t) = u(b,t) = 0,\ \ v(a,t) = v(b,t) = 0$ *as* a *and* b *both approaches* $\mp\infty$. The *exact solution is known as* [7]

$$u(x,t) = \frac{3}{4}sech^2[\frac{1}{2}\sqrt{\frac{1}{2}}(x-\frac{1}{2}t)],\ \ v(x,t) = \frac{3}{4}\sqrt{\frac{1}{2}}\tanh^2[\frac{1}{2}\sqrt{\frac{1}{2}}(x-\frac{1}{2}t)]. \tag{43}$$

The numerical solutions are shown in Figures $3-9$. These solutions are the bell-shaped waves, which agree with the results of [7]. From Figures 3 and 5, we understand that in both cases of u and v, the solutions are a solitary wave pattern. Also, from Figures 4 and 6, we see that for $t \ge 0.5$, the solution starts to bifurcate into three waves. Figures $7, 8$ show both the approximate and exact solutions for both $u(x,t)$ and $v(x,t)$, while Figure 9 shows the absolute error when finding the solution of $u(x,t)$.

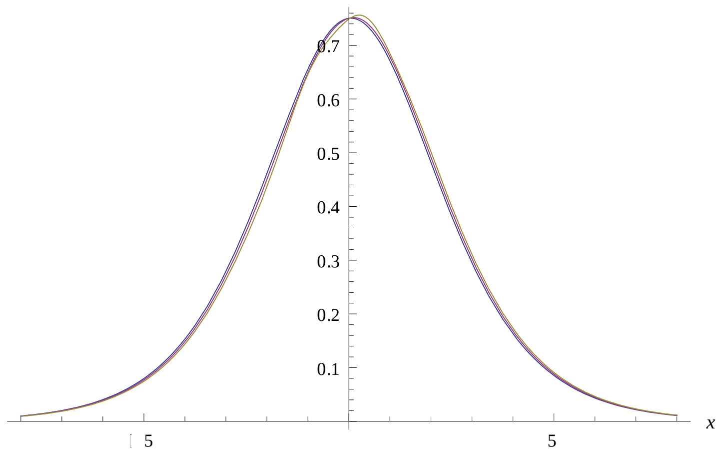

Figure 3.Three profile solutions for $v(x,t)$ *when* $N = 160$, *t =0.1, 0.2, 0.3 and* $\delta t = 0.05$**.**

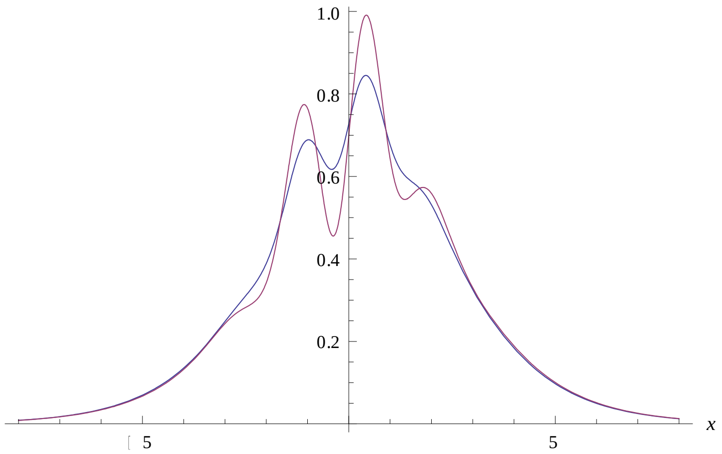

Figure 4. Two profile solutions for $u(x,t)$ when $N = 160$, *t =0.5, 0.6 and* $\delta t = 0.05$**.**

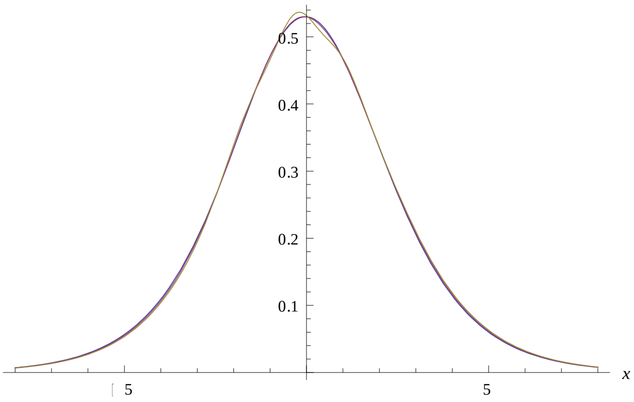

Figure 5. Three profile solutions for $v(x,t)$ when $N=160$, $t=0.1, 0.2, 0.3$ *and* $\delta t = 0.05$.

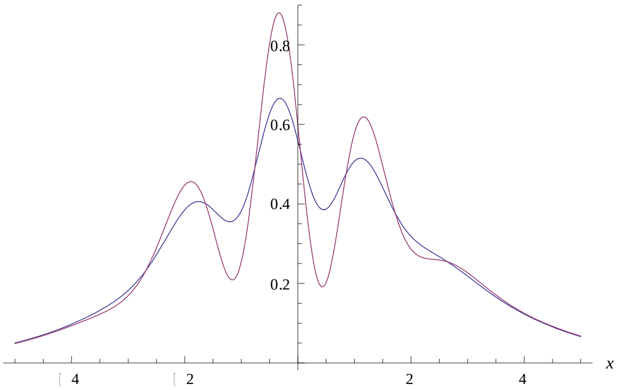

Figure 6. Two profile solutions for $v(x,t)$ when $N=160$, $t=0.5, 0.6$ *and* $\delta t = 0.01$.

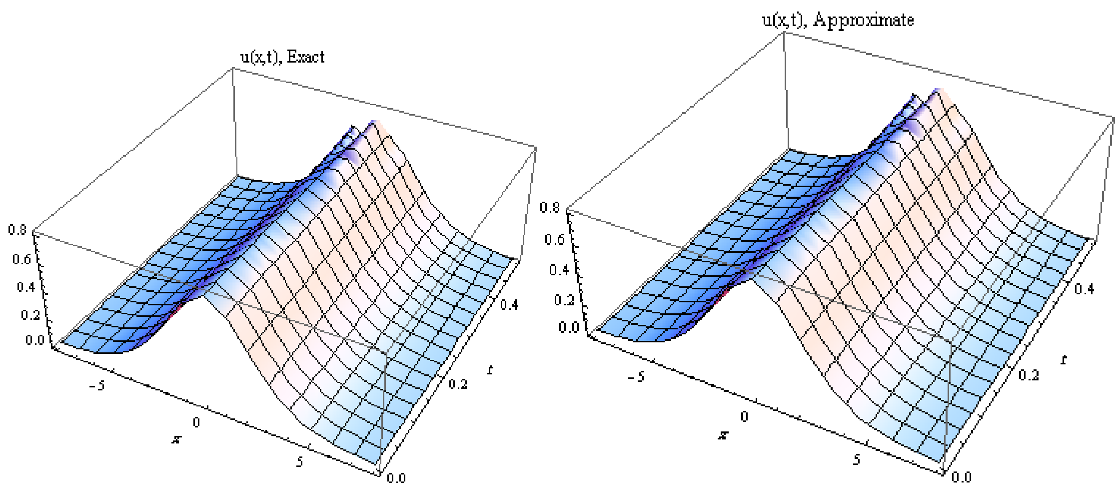

Figure 7. The exact and approximate solutions of $u(x,t)$ in Example 5.2.

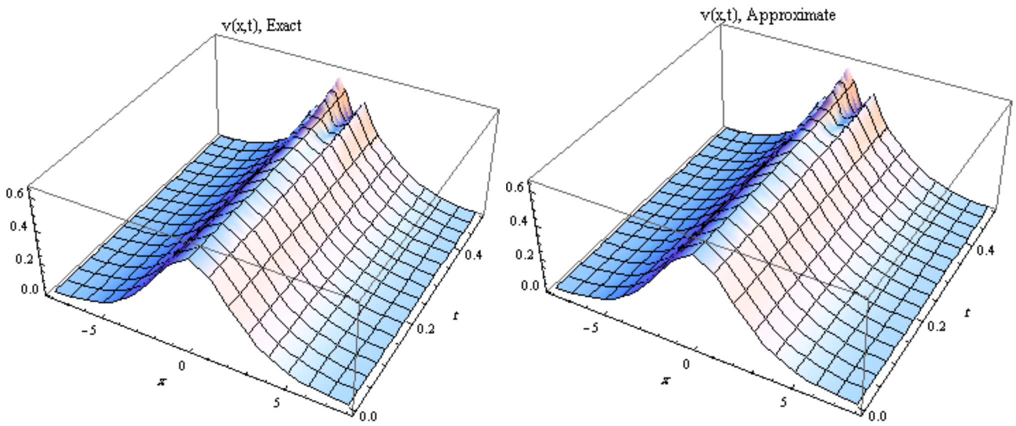

Figure 8. The exact and approximate solutions of $v(x,t)$ in Example 5.2.

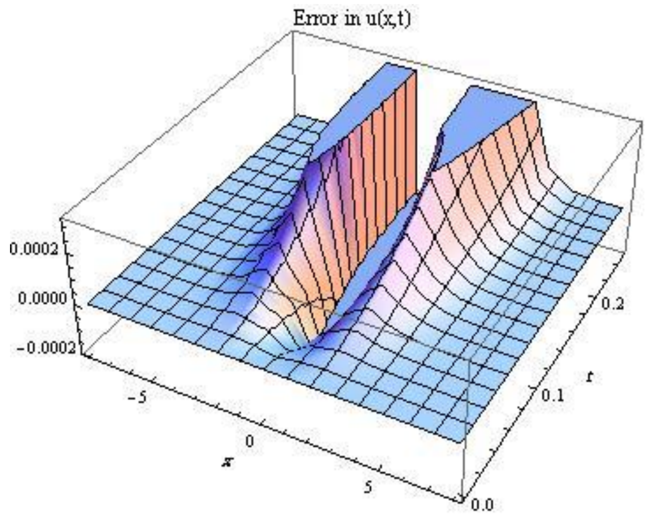

Figure 9. The error in our approximate solution for $u(x,t)$ in Example 5.2.

Table 3. The absolute error in finding both $u(x,t)$ and $v(x,t)$ in Example 5.2 for $t = 0.1$.

x	$\lvert u-\tilde{u}\rvert$	$\lvert v-\tilde{v}\rvert$
–5	2.6530×10^{-6}	2.8135×10^{-4}
–4	1.7980×10^{-5}	1.0097×10^{-3}
–3	9.9222×10^{-5}	3.1410×10^{-3}
–2	2.6240×10^{-4}	7.1015×10^{-3}
–1	1.6752×10^{-4}	7.6237×10^{-3}
0	4.8000×10^{-4}	1.8600×10^{-4}
1	3.9810×10^{-5}	7.1700×10^{-3}
2	1.7891×10^{-4}	6.5813×10^{-3}
3	9.5560×10^{-5}	3.3004×10^{-3}
4	2.5442×10^{-5}	1.1990×10^{-3}
5	4.7766×10^{-6}	3.6223×10^{-4}

6. Conclusions

The fundamental goal of this paper wasto propose an efficient algorithm for the solution of coupled KdV equations. The Sinc-collocation method was described in detail, and implemented to compute a numerical solution to the system in (1). A brief stability analysis was provided, which produced a necessary condition for stability of the method. The efficiency of the method wastested on one example of soliton type, and the accuracy examined in terms of the L_∞, L_2 error norms. The results obtained by the Sinc collocation method were very close to analytical ones, and were found to be more accurate than other numerical schemes [12,17]. The algorithm has been found to be stable, exponentially convergent in space and a reliable numerical method for solving coupled KdV equations.

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Received 28 August 2014
Accepted 24 February 2015