SQU Journal for Science, 2015, **20(2)**, 31-41

A Priori Error Estimates for Mixed Finite Element θ-Schemes for the Wave Equation

Samir Karaa

Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O. Box: 36, Al-Khod 123, Muscat, Sultanate of Oman. Email: skaraa@squ.edu.om.

ABSTRACT: A family of implicit-in-time mixed finite element schemes is presented for the numerical approximation of the acoustic wave equation. The mixed space discretization is based on the displacement form of the wave equation and the time-stepping method employs a three-level one-parameter scheme. A rigorous stability analysis is presented based on energy estimation and sharp stability results are obtained. A convergence analysis is carried out and optimal a priori $L^\infty(L^2)$ error estimates for both displacement and pressure are derived.

Keywords: Energy technique; Error estimation; Mixed finite elements; Wave equation.

تقديرات الخطأ المسبقة لطرق العناصرالمحدودة المختلطة- θ لمعادلة الموجة

سمير القرعة

ملخص: نعرض زمرة من طرق العناصرالمحدودة المختلطة، والضمنية في الزمن، للتقريب العددي لمعادلة الموجة الصوتية. تم تقسيم الفضاء بالاستناد إلى حالة الإزاحة لمعادلة الموجة. يعتمد التقريب الزمني على ثلاثة مستويات مرتبطة بمتغير واحد. تم تحليل استقرارالطرق المستخدمة على أساس تقديرالطاقة والتوصل إلى نتائج دقيقة. كما تم تحليل التقارب وتقدير الخطأ الأولي $L^\infty(L^2)$ بشكل مثالي لكل من الإزاحة والضغط.

كلمات مفتاحية: معادلة الموجة، العناصرالمحدودة المختلطة، تقدير الخطأ، تقنية الطاقة.

1. Introduction

The acoustic wave equation is used to model the effects of wave propagation in heterogeneous media. Solving this equation efficiently is of fundamental importance in many real-life problems. In geophysics, it helps for instance in the interpretation of the seismograph field data and to predict damage patterns due to earthquakes. Using finite element methods for its approximation is attractive because of the ability to handle complex discretizations and design adaptive grid refinement strategies based on error indicators.

Previous attempts on wave simulation by finite elements have used continuous Galerkin methods [1-6], mixed finite element methods [7-13], and discontinuous Galerkin methods [14-17]. In a mixed finite element formulation both displacements and stresses are approximated simultaneously. This approach provides higher-order approximations to the stresses. This property is important in many problems, in particular in modeling boundary controlability of the wave equation [18]. One of the main difficulties of the mixed finite element techniques is the requirement of compatibility of the approximating spaces for convergence and stability.

Given a bounded convex polygonal domain Ω in $\mathbb{R}^m$, $m = 2,3$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, and unit outward normal ν, the general form of the wave equation is

$$\rho \mathbf{u}_{tt} + \nabla\cdot\tilde{\tau} = \mathbf{f} \quad in\ \Omega\times(0,T), \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad on\ \Gamma_D\times(0,T), \tag{2}$$

$$\mathbf{u}\cdot\nu = 0 \quad on\ \Gamma_N\times(0,T), \tag{3}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}^0 \quad in\ \Omega, \tag{4}$$

$$\mathbf{u}_t(\cdot,0) = \mathbf{v}^0 \quad in\ \Omega, \tag{5}$$

where $\mathbf{u}$ is the displacement, ρ is the density, and $\tilde{\tau}$ is the stress tensor given by the generalized Hooke's law $\tilde{\tau} = \lambda(\nabla\cdot\mathbf{u})\tilde{\mathbf{I}} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. Here $\lambda > 0$ and μ are the Lamé coefficients characterizing the material. The function

$\mathbf{f}$ represents a general source term and $\mathbf{u}^0$ and $\mathbf{v}^0$ are initial conditions on displacements and velocities. We assume that $\mathbf{f}$, $\mathbf{u}^0$ and $\mathbf{v}^0$ are sufficiently smooth that there is a unique solution $\mathbf{u} \in C^2((0,T)\times\Omega)$ to (1)-(5), see [19].

The limiting case of (1) with $\mu = 0$ is referred to as the acoustic wave equation, which is

$$\rho\mathbf{u}_{tt} + \nabla\cdot(\lambda(\nabla\cdot\mathbf{u})\tilde{\mathbf{I}}) = \mathbf{f}. \tag{6}$$

It is assumed that ρ and λ are bounded below and above by the positive constants ρ_0, ρ_1, λ_0, and λ_1, respectively. This vector equation is equivalent to the scalar wave equation after making the substitution $p = \lambda\nabla, u$. The mixed method is established by using this relationship, leading to the coupled system

$$\rho\mathbf{u}_{tt} - \nabla p = \mathbf{f} \qquad in\ \Omega\times(0,T), \tag{7}$$

$$\lambda^{-1} p = \nabla\cdot\mathbf{u} \quad in\ \Omega\times(0,T), \tag{8}$$

with the appropriate boundary and initial conditions.

A priori error estimates for solving (7)-(8) were obtained in [7, 8, 9, 10]. In [9], Geveci derived L^∞- in-time, L^2- in-space error bounds for the continuous-in-time mixed finite element approximations of velocity and stress. In [7,8], a priori error estimates were obtained for the mixed finite element approximation of displacement which requires less regularity than was needed in [9]. Stability for a family of discrete-in-time schemes was also demontratred. In [10], an alternative mixed finite element displacement formulation was proposed reducing requirement on the regularity on the displacement variable. For the explicit discrete-in-time problem, stability results were established and error estimates were obtained. The effectiveness of the method analyzed in [10] was demonstrated in [20] by providing simulations using both lowest-order and next-to-lowest-order Raviart-Thomas elements on rectangles [21].

The purpose of this paper is to analyze an implicit time-stepping method combined with the mixed finite element discretization proposed in [10]. We prove the stability of the proposed method by using energy estimation, and show in particular that it conserves certain energy. We also investigate the convergence of the method and prove optimal a priori $L^\infty(L^2)$ error estimates for both displacement and pressure. The rest of the paper is organized as follows. In Sections 2 and 3, we introduce notations and describe the weak formulation of the problem. The fully discrete mixed finite element method is presented in Section 4. Stability results are established in Section 4 and optimal a priori error estimates are obtained in Section 5. Conclusions are given in the last section.

2. Notation

We shall use the following inner products and norms in this paper. The L^2-inner product over Ω is defined by

$$(u,v) = \int_\Omega uv\,d\Omega,$$

inducing the L^2- norm over Ω, $\|v\|_{L^2(\Omega)} = (v,v)^{1/2}$. The inner product over the boundary $\partial\Omega$ is denoted by

$$\langle u,v\rangle = \int_{\partial\Omega} uv\,d\Omega$$

for $u,\ v \in H^{\frac{1}{2}+\varepsilon}(\Omega)$ with $\varepsilon > 0$. We introduce the time-space norm:

$$\|v\|_{L^2(0,T;L^2(\Omega))} = \|v\|_{L^2(L^2)} = \left(\int_0^T \|v\|^2_{L^2(\Omega)}\,dt\right)^{\frac{1}{2}}.$$

The time-space norm $\|\cdot\|_{L^\infty(L^2)}$ is similarly defined. In addition to the L^2 spaces, we use the standard Sobolev space for mixed methods:

$$\mathbf{H}(\Omega,div) = \{\mathbf{v} : \mathbf{v}\in(L^2(\Omega))^m, \nabla\cdot\mathbf{v}\in L^2(\Omega)\},$$

with associated norm

$$\|\mathbf{v}\|_{\mathbf{H}(\Omega,div)} = \|\mathbf{v}\|_{L^2(\Omega)} + \|\nabla\cdot\mathbf{v}\|_{L^2(\Omega)},$$

where

$$\|\mathbf{v}\|_{L^2(\Omega)} = \left(\sum_{i=1}^m \|v_i\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

For the time discretization, we adopt the following notation. Let N be a positive integer, $\Delta t = T/N$, and $t^n = n\Delta t$. For any function v of time, let v^n denote $v(t^n)$. We shall use this notation for functions defined for all times as well as those defined only at discrete times. Set

$$v^{n+\frac{1}{2}} = \frac{1}{2}\left(v^{n+1} + v^n\right),$$

$$\bar{\partial}_t v^{n+\frac{1}{2}} = \frac{1}{\Delta t}\left(v^{n+1} - v^n\right),$$

$$\bar{\partial}_t v^n = \frac{1}{2\Delta t}\left(v^{n+1} - v^{n-1}\right),$$

$$\bar{\partial}_{tt} v^n = \frac{1}{\Delta t^2}\left(v^{n+1} - 2v^n + v^{n-1}\right),$$

$$v^{n;\theta} = \theta v^{n+1} + (1-2\theta)v^n + \theta v^{n-1},$$

where $0 \le \theta \le 1$. We also define the discrete l^∞- norm for time-discrete functions by

$$\| v \|_{l^\infty_{\Delta t}(0,T;L^2(\Omega))} = \| v \|_{l^\infty(L^2)} = \max_{0\le n\le N} \| v^n \|_{L^2(\Omega)} .$$

3. Weak Formulation

The finite element approximation of the wave problem is based on its weak formulation which is derived in the usual manner. Integrating by parts and using the data on the boundary of Ω, we obtain the weak formulation [10], For any $t \ge 0$, find $(\mathbf{u}(t), p(t)) \in \mathbf{V}\times W$ such that

$$(\mathbf{u}(0), \mathbf{v}) = (\mathbf{u}^0, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{9}$$

$$(\mathbf{u}_t(0), \mathbf{v}) = (\mathbf{v}^0, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{10}$$

$$(\lambda^{-1} p(0), w) = (\nabla\cdot\mathbf{u}^0, w) \quad \forall w \in W, \tag{11}$$

$$(\rho \mathbf{u}_{tt}(t), \mathbf{v}) + (p(t), \nabla\cdot\mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \quad t > 0, \tag{12}$$

$$(\lambda^{-1} p(t), w) - (\nabla\cdot\mathbf{u}(t), w) = 0 \qquad \forall w \in W, \quad t > 0, \tag{13}$$

where $\mathbf{V}$ and W are given by

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}(\Omega, div) : \mathbf{v}\cdot\nu|_{\Gamma_N} = 0\},$$

$$W = H^{\frac{1}{2}+\varepsilon}(\Omega) \text{ for any } \varepsilon > 0.$$

The present formulation requires less regularity on the displacement than standard approaches. For instance in [7, 8] it is necessary that $\nabla p \in \mathbf{H}(\Omega, div)$ so that $\nabla\cdot\mathbf{u} \in H^2(\Omega)$. Here, it is only required that $\nabla\cdot\mathbf{u} \in H^{\frac{1}{2}}$, and it can be verified that the solution $\mathbf{u}$ of problem (1)-(5) with $p = \lambda\nabla\cdot\mathbf{u}$ is a solution to (12)-(13), see [10].

Differentiate (13) with respect to time to obtain

$$(\lambda^{-1} p_t, w) - (\nabla\cdot\mathbf{u}_t, w) = 0 \qquad \forall w \in W. \tag{14}$$

We next assume $\mathbf{f} = 0$ and choose $\mathbf{v} = \mathbf{u}_t$ and $w = p$ in (12) and (14), respectively, so that

$$(\rho \mathbf{u}_{tt}, \mathbf{u}_t) + (p, \nabla\cdot\mathbf{u}_t) = 0, \tag{15}$$

$$(\lambda^{-1} p_t, p) - (\nabla\cdot\mathbf{u}_t, p) = 0. \tag{16}$$

By adding the two equations, we find that

$$(\rho \mathbf{u}_{tt}, \mathbf{u}_t) + (\lambda^{-1} p_t, p) = 0, \tag{17}$$

or

$$\frac{1}{2}\frac{d}{dt}\left\| \rho^{\frac{1}{2}} \mathbf{u}_t \right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\left\| \lambda^{-\frac{1}{2}} p \right\|^2_{L^2(\Omega)} = 0.$$

Thus, in the absence of forcing, the (continuous) energy

$$\frac{1}{2}\left\| \rho^{\frac{1}{2}} \mathbf{u}_t \right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\| \lambda^{-\frac{1}{2}} p \right\|^2_{L^2(\Omega)} \tag{18}$$

is conserved for all time. It will be shown that a similar form of energy is conserved by the numerical solution of the wave problem.

4. Finite Element Approximation

For the finite element approximation, we let $\{\mathcal{E}_h\}_{h>0}$ be a quasi-uniform family of finite element partitions of Ω, where h is the maximum element diameter. Let $\mathbf{V}_h \times W_h$ be any of the usual mixed finite element approximating subspaces of $\mathbf{V}\times W$, that is, the Raviart-Thomas-Nedelec spaces [21, 22], Brezzi-Douglas-Marini spaces [23], or

Brezzi-Douglas-Fortin-Marini spaces [24]. For each of these mixed spaces there is a projection $\Pi_h : \mathbf{H}(\Omega, div) \to \mathbf{V}_h$ such that for any $\mathbf{z} \in \mathbf{H}(\Omega, div)$

$$(\nabla \cdot \Pi_h \mathbf{z}, w) = (\nabla \cdot \mathbf{z}, w) \quad \forall w \in W_h. \tag{19}$$

We have the property that, if $\mathbf{z} \in \mathbf{H}(\Omega, div) \cap \mathbf{H}^k(\Omega)$, then

$$\| \Pi_h \mathbf{z} - \mathbf{z} \|_0 \le C h^j \| \mathbf{z} \|_j, \quad 1 \le j \le k, \tag{20}$$

where k is associated with the degree of polynomial and $\| \cdot \|_s$ is the standard Sobolev norm on $(H^s(\Omega))^m$. Here and in what follows, C is a generic positive constant which is independent of h and Δt.

For $\phi \in W$, we denote by $P_h \phi$ the L^2-projection of ϕ onto W_h defined by requiring that

$$(P_h \phi, w) = (\phi, w) \quad \forall w \in W_h. \tag{21}$$

If $\phi \in W \cap H^k(\Omega)$, then we also have

$$\| P_h \phi - \phi \|_s \le C h^{j-s} \| \phi \|_j, \quad 0 \le s \le k, \quad 0 \le j \le k. \tag{22}$$

The semidiscrete mixed finite element approximation to $(\mathbf{u}(t), p(t))$ is to seek $(\mathbf{U}(t), P(t)) \in \mathbf{V}_h \times W_h$ satisfying

$$(\mathbf{U}(0), \mathbf{v}) = (\Pi_h \mathbf{u}^0, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{23}$$

$$(\mathbf{U}_t(0), \mathbf{v}) = (\Pi_h \mathbf{v}^0, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{24}$$

$$(P(0), w) = (p(0), w) \quad \forall w \in W_h, \tag{25}$$

$$(\rho \mathbf{U}_{tt}(t), \mathbf{v}) + (P(t), \nabla \cdot \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h, \quad t > 0, \tag{26}$$

$$(\lambda^{-1} P(t), w) - (\nabla \cdot \mathbf{U}(t), w) = 0 \qquad \forall w \in W_h, \quad t > 0. \tag{27}$$

Existence and uniqueness of a solution $(\mathbf{U}(t), P(t))$ to the variational problem (23)-(27) is shown in [10].

The fully discrete mixed finite element θ-scheme is then defined by finding a sequence of pairs $(\mathbf{U}^n, P^n) \in \mathbf{V}_h \times W_h$, $0 \le n \le N$, such that

$$(\mathbf{U}^0, \mathbf{v}) = (\Pi_h \mathbf{u}^0, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{28}$$

$$(P^0, w) = (p^0, w) \quad \forall w \in W_h, \tag{29}$$

$$\left(\rho \bar{\partial}_t \mathbf{U}^{\frac{1}{2}}, \mathbf{v} \right) + \theta^2 \Delta t \left(\bar{\partial}_t P^{\frac{1}{2}}, \nabla \cdot \mathbf{v} \right) + \frac{\Delta t}{2} (P^0, \nabla \cdot \mathbf{v}) = \left(\frac{\Delta t}{2} \mathbf{f}^0 + \theta \Delta t^2 \bar{\partial}_t \mathbf{f}^{\frac{1}{2}}, \mathbf{v} \right) + \left(\rho \Pi_h \mathbf{v}^0, \mathbf{v} \right) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{30}$$

$$(\rho \bar{\partial}_{tt} \mathbf{U}^n, \mathbf{v}) + (P^{n;\theta}, \nabla \cdot \mathbf{v}) = (\mathbf{f}^{n;\theta}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{31}$$

$$(\lambda^{-1} P^{n+1/2}, w) - (\nabla \cdot \mathbf{U}^{n+1/2}, w) = 0 \quad \forall w \in W_h. \tag{32}$$

Equation (30) is derived from the following expansion:

$$\mathbf{u}^1 = \mathbf{u}^0 + \Delta t \mathbf{v}^0 + \Delta t^2 \left[\theta \mathbf{u}_{tt}^1 + \left(\frac{1}{2} - \theta \right) \mathbf{u}_{tt}^0 \right] + O(\Delta t^3).$$

The present θ- scheme is explicit in time if $\theta = 0$ and implicit otherwise. The existence and uniqueness of a solution to the resulting linear system for a nonzero value of θ follows from the unisolvancy of the mixed formulation of the following elliptic problem:

$$\nabla \cdot (\lambda \nabla \phi) + \frac{1}{\theta \Delta t^2} \rho \phi = 0 \qquad in\ \Omega,$$

$$\phi = 0 \qquad on\ \partial\Omega.$$

The explicit case has been considered in [10]. As expected from an explicit scheme, the method is conditionally stable. As a stability constraint, it requires the choice of

$$\Delta t = O(h).$$

In the next sections, stability and convergence properties of the proposed θ- scheme are analyzed.

5. Stability Analysis

We derive sharp stability bounds based on the energy technique and show that the proposed scheme conserves certain energy. We consider (31) and (32) for the homogeneous case

$$(\rho \bar{\partial}_{tt} \mathbf{U}^n, \mathbf{v}) + (P^{n;\theta}, \nabla \cdot \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{33}$$

$$(\lambda^{-1} P^{n+1/2}, w) - (\nabla \cdot \mathbf{U}^{n+1/2}, w) = 0 \qquad \forall w \in W_h. \tag{34}$$

We will make use of the *inverse assumption*, which states that there exists a constant C_0 independent of h, such that

$$\| \nabla \cdot \phi \|_{L^2(\Omega)} \le C_0 h^{-1} \| \phi \|_{L^2(\Omega)} \tag{35}$$

for all $\phi \in W_h$. The following stability result holds.

Theorem 1 *The fully discrete scheme* (28)-(32) *is stable if*

$$\Delta t^2\left(\frac{1}{4}-\theta\right)\frac{C_0^2\lambda_1}{h^2\rho_0}\le 1, \tag{36}$$

and conserves the discrete energy

$$E_h^{n+\frac{1}{2}}=\frac{1}{2}\left[\|\rho^{\frac{1}{2}}\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}\|^2+\Delta t^2\left(\theta-\frac{1}{4}\right)\|\lambda^{-\frac{1}{2}}\bar{\partial}_t P^{n+\frac{1}{2}}\|^2+\|\lambda^{-\frac{1}{2}}P^{n+\frac{1}{2}}\|^2\right]. \tag{37}$$

The scheme is unconditionally stable if $\theta\ge 1/4$.

Proof. If we subtract (34) from itself, with $n+1/2$ replaced by $n-1/2$, we find that

$$(\lambda^{-1}(P^{n+1}-P^{n-1},w)-(\nabla\cdot(U^{n+1}-U^{n-1}),w)=0. \tag{38}$$

As (33) holds for all $\mathbf{v}\in\mathbf{V}_h$ and (38) holds for all $w\in W_h$, we choose $\mathbf{v}=\bar{\partial}_t\mathbf{U}^n$ and $w=\dfrac{P^{n;\theta}}{2\Delta t}$ so that

$$(\rho\bar{\partial}_{tt}\mathbf{U}^n,\bar{\partial}_t\mathbf{U}^n)+(P^{n;\theta},\nabla\cdot\bar{\partial}_t\mathbf{U}^n)=0, \tag{39}$$

$$(\lambda^{-1}\bar{\partial}_t P^n,P^{n;\theta})-(\nabla\cdot\bar{\partial}_t\mathbf{U}^n,P^{n;\theta})=0. \tag{40}$$

By adding (39) and (40) we obtain

$$(\rho\bar{\partial}_{tt}\mathbf{U}^n,\bar{\partial}_t\mathbf{U}^n)+(\lambda^{-1}\bar{\partial}_t P^n,P^{n;\theta})=0. \tag{41}$$

Note that

$$\begin{aligned}P^{n;\theta}&=\Delta t^2\theta\bar{\partial}_{tt}P^n+P^n\\&=\Delta t^2\left(\theta-\frac{1}{4}\right)\bar{\partial}_{tt}P^n+\frac{1}{2}\left(P^{n+\frac{1}{2}}+P^{n-\frac{1}{2}}\right).\end{aligned} \tag{42}$$

Hence, (41) can be rewritten as

$$(\rho\bar{\partial}_{tt}\mathbf{U}^n,\bar{\partial}_t\mathbf{U}^n)+\Delta t^2\left(\theta-\frac{1}{4}\right)(\lambda^{-1}\bar{\partial}_{tt}P^n,\bar{\partial}_t P^n)+\frac{1}{2}\left(\lambda^{-1}(P^{n+\frac{1}{2}}+P^{n-\frac{1}{2}}),\bar{\partial}_t P^n\right)=0. \tag{43}$$

Using that

$$\bar{\partial}_t\mathbf{U}^n=\frac{\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}+\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}}}{2},\qquad \bar{\partial}_{tt}\mathbf{U}^n=\frac{\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}-\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}}}{\Delta t},$$

we deduce that

$$\begin{aligned}(\rho\bar{\partial}_{tt}\mathbf{U}^n,\bar{\partial}_t\mathbf{U}^n)&=\frac{1}{2\Delta t}(\rho\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}-\rho\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}},\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}+\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}})\\&=\frac{1}{2\Delta t}\left[(\rho\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}},\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}})-(\rho\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}},\bar{\partial}_t\mathbf{U}^{n-\frac{1}{2}})\right],\end{aligned}$$

and similarly

$$(\lambda^{-1}\bar{\partial}_{tt}P^n,\bar{\partial}_t P^n)=\frac{1}{2\Delta t}\left[(\lambda^{-1}\bar{\partial}_t P^{n+\frac{1}{2}},\bar{\partial}_t P^{n+\frac{1}{2}})-(\lambda^{-1}\bar{\partial}_t P^{n-\frac{1}{2}},\bar{\partial}_t P^{n-\frac{1}{2}})\right].$$

We also have

$$\begin{aligned}\left(\lambda^{-1}(P^{n+\frac{1}{2}}+P^{n-\frac{1}{2}}),\bar{\partial}_t P^n\right)&=\frac{1}{\Delta t}(\lambda^{-1}P^{n+\frac{1}{2}}+\lambda^{-1}P^{n-\frac{1}{2}},P^{n+\frac{1}{2}}-P^{n-\frac{1}{2}})\\&=\frac{1}{\Delta t}\left[(\lambda^{-1}P^{n+\frac{1}{2}},P^{n+\frac{1}{2}})-(\lambda^{-1}P^{n-\frac{1}{2}},P^{n-\frac{1}{2}})\right].\end{aligned}$$

Hence, (43) is equivalent to

$$\frac{1}{\Delta t}\left(E_h^{n+\frac{1}{2}}-E_h^{n-\frac{1}{2}}\right)=0,$$

where $E_h^{n+\frac{1}{2}}$ is the quantity defined by (37). This relation indicates that $E_h^{n+\frac{1}{2}}$ is conserved for all time, which guarantees the stability of the scheme if and only if $E_h^{n+\frac{1}{2}}$ defines a positive energy. A sufficient condition is that

$$\left\|\rho^{\frac{1}{2}}\bar{\partial}_t\mathbf{U}^{n+\frac{1}{2}}\right\|^2+\Delta t^2\left(\theta-\frac{1}{4}\right)\left\|\lambda^{-\frac{1}{2}}\bar{\partial}_t P^{n+\frac{1}{2}}\right\|^2\ge 0$$

for all $n \geq 0$. Clearly, the scheme is unconditionally stable when $\theta \geq 1/4$. Now, using the Cauchy-Schwarz inequality and the inverse assumption (35), we obtain

$$\left(\lambda^{-1}\bar{\partial}_t P^{n+\frac{1}{2}}, w\right) = \left(\nabla \cdot \bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}, w\right)$$

$$\leq \left\|\nabla \cdot \bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}\right\|_{L^2(\Omega)} \| w \|_{L^2(\Omega)}$$

$$\leq \frac{C_0}{h}\left\|\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}\right\|_{L^2(\Omega)} \| w \|_{L^2(\Omega)}.$$

By setting $w = \bar{\partial}_t P^{n+\frac{1}{2}}$, we see that

$$\left\|\lambda^{-\frac{1}{2}}\bar{\partial}_t P^{n+\frac{1}{2}}\right\|^2_{L^2(\Omega)} \leq \frac{C_0}{h}\left\|\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\bar{\partial}_t P^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}$$

$$\leq \frac{C_0\lambda_1^{\frac{1}{2}}}{h\rho_0^{\frac{1}{2}}}\left\|\rho^{\frac{1}{2}}\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\lambda^{-\frac{1}{2}}\bar{\partial}_t P^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}$$

or

$$\left\|\lambda^{-\frac{1}{2}}\bar{\partial}_t P^{n+\frac{1}{2}}\right\|_{L^2(\Omega)} \leq \frac{C_0\lambda_1^{\frac{1}{2}}}{h\rho_0^{\frac{1}{2}}}\left\|\rho^{\frac{1}{2}}\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}.$$

Hence, a sufficient condition for stability is given by

$$\| \rho^{\frac{1}{2}}\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}} \|^2 + \Delta t^2\left(\theta - \frac{1}{4}\right)\frac{C_0^2\lambda_1}{h^2\rho_0}\| \rho^{\frac{1}{2}}\bar{\partial}_t \mathbf{U}^{n+\frac{1}{2}} \|^2 \geq 0,$$

which completes the proof. □

The case with $\theta = 1/4$ is interesting because the form of the discrete energy in this case is similar to that of the continuous problem. In addition, one can verify that the time truncation error is minimized over the set of all $\theta \geq 1/4$ when $\theta = 1/4$.

6. Convergence Analysis

In this section, we prove optimal convergence of the fully discrete finite element solution in the $L^\infty(L^2)$ norm. Some of the techniques used in the proofs can be found in previous works [25, 26]. In order to estimate the errors in the finite element approximation, we define the auxiliary functions

$$\chi^n = \mathbf{U}^n - \Pi_h \mathbf{u}^n, \quad \eta^n = \mathbf{u}^n - \Pi_h \mathbf{u}^n, \qquad \xi^n = P^n - P_h p^n, \qquad \zeta = p^n - P_h p^n,$$

where Π_h and P_h are defined in Section 4. From (12-13) and (31-32), and the properties of the projections Π_h and P_h, we arrive at

$$(\rho\bar{\partial}_{tt}\chi^n, v) + (\xi^{n;\theta}, \nabla \cdot \mathbf{v}) = (\rho\bar{\partial}_{tt}\eta^n, \mathbf{v}) + (\mathbf{r}^n, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad n \geq 1, \tag{44}$$

$$(\lambda^{-1}\xi^{n+1/2}, w) - (\nabla \cdot \chi^{n+1/2}, w) = (\lambda^{-1}\zeta^{n+1/2}, w) \qquad \forall w \in W_h, \quad n \geq 0, \tag{45}$$

where $\mathbf{r}^n = \rho(\mathbf{u}_{tt}^{n;\theta} - \bar{\partial}_{tt}\mathbf{u}^n)$. Another equation has to be derived for the initial errors χ^1 and ξ^1. Consider (12) at $n = 0$ and $n = 1$, respectively, and subtract the resulting equations so that

$$\left(\rho\bar{\partial}_t \mathbf{u}_{tt}^{\frac{1}{2}}, \mathbf{v}\right) + \left(\bar{\partial}_t p^{\frac{1}{2}}, \nabla \cdot \mathbf{v}\right) = \left(\bar{\partial}_t \mathbf{f}^{\frac{1}{2}}, \mathbf{v}\right). \tag{46}$$

A use of Taylor's formula with integral remainder yields

$$\bar{\partial}_t \mathbf{u}^{\frac{1}{2}} = \mathbf{v}^0 + \frac{\Delta t}{2}\mathbf{u}_{tt}^0 + \frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t - t)^2\frac{\partial^3 \mathbf{u}}{\partial t^3}(t)dt. \tag{47}$$

Using (46) and (47), we readily obtain

$$\left(\rho\bar{\partial}_t \mathbf{u}^{\frac{1}{2}}, \mathbf{v}\right) + \theta\Delta t^2\left(\bar{\partial}_t p^{\frac{1}{2}}, \nabla \cdot \mathbf{v}\right) = \theta\Delta t^2\left(\bar{\partial}_t \mathbf{f}^{\frac{1}{2}}, \mathbf{v}\right) - \theta\Delta t^2\left(\rho\bar{\partial}_t \mathbf{u}_{tt}^{\frac{1}{2}}, \mathbf{v}\right)$$

$$+(\rho\mathbf{v}^0,\mathbf{v})+\frac{\Delta t}{2}(\rho\mathbf{u}_{tt}^0,\mathbf{v}) \ +\frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t-t)^2\left(\rho\frac{\partial^3\mathbf{u}}{\partial t^3},\mathbf{v}\right)dt. \tag{48}$$

Subtracting (48) from (30) and taking into account (29) and (12) we arrive at

$$\begin{aligned}\left(\rho\bar{\partial}_t\chi^{\frac{1}{2}},\mathbf{v}\right)+\theta\Delta t^2\left(\bar{\partial}_t\xi^{\frac{1}{2}},\nabla\cdot\mathbf{v}\right) \ +\frac{\Delta t}{2}\left(\xi^0,\nabla\cdot\mathbf{v}\right)=\left(\rho\bar{\partial}_t\eta^{\frac{1}{2}},\mathbf{v}\right)+\left(\rho(\Pi_h\mathbf{v}^0-\mathbf{v}^0),\mathbf{v}\right)\\ +\theta\Delta t^2\left(\rho\bar{\partial}_t\mathbf{u}_{tt}^{\frac{1}{2}},\mathbf{v}\right)-\frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t-t)^2\left(\rho\frac{\partial^3\mathbf{u}}{\partial t^3},\mathbf{v}\right)dt.\end{aligned} \tag{49}$$

Note that $\xi^0=0$ and $\chi^0=0$. We now state and prove our convergence result.

Theorem 2 *If* $\mathbf{u}\in L^\infty(\mathbf{H}(\Omega;\mathrm{d}iv))$, $\frac{\partial^3\mathbf{u}}{\partial t^3}\in L^1(\mathbf{L}^2(\Omega))$, $\frac{\partial^4\mathbf{u}}{\partial t^4}\in L^\infty(\mathbf{L}^2(\Omega))$, *and* $p\in L^\infty(L^2(\Omega))$, *then for* $\{\mathbf{U}^n,P^n\}$ *defined by (28)-(32) there exists a constant* C *independent of* h *and* Δt *such that if*

$$\Delta t^2\left(\frac{1}{4}-\theta\right)\frac{\lambda_1 C_0^2}{\rho_0 h^2}<\frac{1}{2}, \tag{50}$$

then the following a priori error estimate holds:

$$\begin{aligned}\left\|\rho^{\frac{1}{2}}(\mathbf{u}-\mathbf{U})\right\|_{l^\infty(L^2)}+\left\|\lambda^{-\frac{1}{2}}(p-P)\right\|_{l^\infty(L^2)}\\ \le C(h^r+\Delta t^2)\left(\|\mathbf{u}\|_{L^\infty(H^r)}+\left\|\frac{\partial^3\mathbf{u}}{\partial t^3}\right\|_{L^\infty(L^2)}+\|p\|_{L^\infty(L^2)}\right),\end{aligned} \tag{51}$$

where r *is associated with the degree of the finite element polynomial.*

Proof. We first rearrange (44) in the form

$$(\rho\bar{\partial}_{tt}\chi^n,\mathbf{v})+\Delta t^2\left(\theta-\frac{1}{4}\right)(\bar{\partial}_{tt}\xi^n,\nabla\cdot\mathbf{v})+\frac{1}{2}\left(\xi^{n+\frac{1}{2}}+\xi^{n-\frac{1}{2}},\nabla\cdot\mathbf{v}\right)=(\rho\bar{\partial}_{tt}\eta^n,v)+(\mathbf{r}^n,\mathbf{v}). \tag{52}$$

Summing over time levels and multiplying through by Δt yields

$$\begin{aligned}(\rho\bar{\partial}_t\chi^{n+\frac{1}{2}}-\rho\bar{\partial}_t\quad\chi^{\frac{1}{2}},\mathbf{v})+\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\bar{\partial}_t\xi^{n+\frac{1}{2}}-\bar{\partial}_t\xi^{\frac{1}{2}},\nabla\cdot\mathbf{v}\right)\\ +\frac{\Delta t}{2}\sum_{i=1}^n\left(\xi^{i+\frac{1}{2}}+\xi^{i-\frac{1}{2}},\nabla\cdot\mathbf{v}\right)=\left(\rho\bar{\partial}_t\eta^{n+\frac{1}{2}}-\rho\bar{\partial}_t\eta^{\frac{1}{2}},\mathbf{v}\right)+\left(\Delta t\sum_{i=1}^n\mathbf{r}^i,v\right).\end{aligned} \tag{53}$$

Upon defining

$$\phi^0=0,\qquad \phi^n=\Delta t\sum_{i=0}^{n-1}\xi^{i+\frac{1}{2}},$$

we verify that

$$\phi^{n+\frac{1}{2}}=\frac{\Delta t}{2}\xi^{\frac{1}{2}}+\frac{\Delta t}{2}\sum_{i=1}^n\left(\xi^{i+\frac{1}{2}}+\xi^{i-\frac{1}{2}}\right).$$

Taking into account (49) and that $\bar{\partial}_t\xi^{\frac{1}{2}}=\frac{2}{\Delta t}\xi^{\frac{1}{2}}$, (53) becomes

$$(\rho\bar{\partial}_t\chi^{n+\frac{1}{2}},\mathbf{v})+\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\bar{\partial}_t\xi^{n+\frac{1}{2}},\nabla\cdot\mathbf{v}\right)+\left(\phi^{n+\frac{1}{2}},\nabla\cdot\mathbf{v}\right)=\left(\rho\bar{\partial}_t\eta^{n+\frac{1}{2}},\mathbf{v}\right)+\left(\mathbf{R}^n,\mathbf{v}\right), \tag{54}$$

where

$$\mathbf{R}^n=\Delta t\sum_{i=1}^n\mathbf{r}^i+\rho(\Pi_h\mathbf{v}^0-\mathbf{v}^0)+\theta\Delta t^2\rho\bar{\partial}_t\mathbf{u}_{tt}^{\frac{1}{2}}-\frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t-t)^2\rho\frac{\partial^3\mathbf{u}}{\partial t^3}dt.$$

Since $\bar{\partial}_t\phi^{n+\frac{1}{2}}=\xi^{n+\frac{1}{2}}$, (45) reads

$$\left(\lambda^{-1}\bar{\partial}_t\phi^{n+\frac{1}{2}},w\right)-\left(\nabla\cdot\chi^{n+\frac{1}{2}},w\right)=\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},w\right). \tag{55}$$

Choosing $v=\chi^{n+\frac{1}{2}}$ and $w=\phi^{n+\frac{1}{2}}$ in (54) and (55), respectively, and adding the resulting equations, we arrive at

$$
\begin{aligned}
(\rho\bar{\partial}_t\chi^{n+\frac{1}{2}},\chi^{n+\frac{1}{2}})+\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\bar{\partial}_t\xi^{n+\frac{1}{2}},\nabla\cdot\chi^{n+\frac{1}{2}}\right)+\left(\lambda^{-1}\bar{\partial}_t\phi^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\right)\\
=\left(\rho\bar{\partial}_t\eta^{n+\frac{1}{2}},\chi^{n+\frac{1}{2}}\right)+\left(\mathbf{R}^n,\chi^{n+\frac{1}{2}}\right)+\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\right).
\end{aligned}
\tag{56}
$$

Again, we choose $w=\bar{\partial}_t\xi^{n+\frac{1}{2}}$ in (45) so that

$$
\left(\bar{\partial}_t\xi^{n+\frac{1}{2}},\nabla\cdot\chi^{n+\frac{1}{2}}\right)=\left(\lambda^{-1}\xi^{n+\frac{1}{2}},\bar{\partial}_t\xi^{n+\frac{1}{2}}\right)-\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\bar{\partial}_t\xi^{n+\frac{1}{2}}\right).
$$

Substitution into (56) yields

$$
\begin{aligned}
(\rho\bar{\partial}_t\chi^{n+\frac{1}{2}},\chi^{n+\frac{1}{2}})+\quad&\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\lambda^{-1}\bar{\partial}_t\xi^{n+\frac{1}{2}},\xi^{n+\frac{1}{2}}\right)+\left(\lambda^{-1}\bar{\partial}_t\phi^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\right)\\
&=\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\bar{\partial}_t\xi^{n+\frac{1}{2}}\right)+\left(\rho\bar{\partial}_t\eta^{n+\frac{1}{2}},\chi^{n+\frac{1}{2}}\right)\\
&\quad+\left(\mathbf{R}^n,\chi^{n+\frac{1}{2}}\right)+\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\right).
\end{aligned}
\tag{57}
$$

The terms on the right-hand side of (57) are bounded using Cauchy-Schwarz inequality as

$$
\begin{aligned}
\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\bar{\partial}_t\xi^{n+\frac{1}{2}}\right)&\le\left\|\lambda^{-\frac{1}{2}}\zeta^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\lambda^{-\frac{1}{2}}\bar{\partial}_t\xi^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\\
\left(\rho\bar{\partial}_t\eta^{n+\frac{1}{2}},\chi^{n+\frac{1}{2}}\right)&\le\left\|\rho\bar{\partial}_t\eta^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\chi^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\\
\left(\mathbf{R}^n,\chi^{n+\frac{1}{2}}\right)&\le\left\|\mathbf{R}^n\right\|_{L^2(\Omega)}\left\|\chi^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\\
\left(\lambda^{-1}\zeta^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\right)&\le\left\|\lambda^{-\frac{1}{2}}\zeta^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\lambda^{-\frac{1}{2}}\phi^{n+\frac{1}{2}}\right\|_{L^2(\Omega)}.
\end{aligned}
$$

We now distinguish the cases where $\theta\ge\frac{1}{4}$ and $\theta<\frac{1}{4}$. In the first case, we sum on (57) over time levels and multiply through by $2\Delta t$. This results in

$$
\begin{aligned}
&\left\|\rho^{\frac{1}{2}}\chi^{n+1}\right\|^2_{L^2(\Omega)}-\left\|\rho^{\frac{1}{2}}\chi^{0}\right\|^2_{L^2(\Omega)}+\left\|\lambda^{-\frac{1}{2}}\phi^{n+1}\right\|^2_{L^2(\Omega)}-\left\|\lambda^{-\frac{1}{2}}\phi^{0}\right\|^2_{L^2(\Omega)}\\
&\quad+\Delta t^2\left(\theta-\frac{1}{4}\right)\left(\left\|\lambda^{-\frac{1}{2}}\xi^{n+1}\right\|^2_{L^2(\Omega)}-\left\|\lambda^{-\frac{1}{2}}\xi^{0}\right\|^2_{L^2(\Omega)}\right)\\
&\le 2\Delta t^2\left(\theta-\frac{1}{4}\right)\sum_{i=0}^{n}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\left(\left\|\lambda^{-\frac{1}{2}}\xi^{i+1}\right\|_{L^2(\Omega)}+\left\|\lambda^{-\frac{1}{2}}\xi^{i}\right\|_{L^2(\Omega)}\right)\\
&\quad+2\Delta t\sum_{i=0}^{n}\left\|\chi^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\left(\left\|\rho\bar{\partial}_t\eta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}+\left\|\mathbf{R}^i\right\|_{L^2(\Omega)}\right)\\
&\quad+2\Delta t\sum_{i=0}^{n}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\lambda^{-\frac{1}{2}}\phi^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}.
\end{aligned}
\tag{58}
$$

Since $\left\|\lambda^{-\frac{1}{2}}\xi^{i}\right\|_{L^2(\Omega)}\le\left\|\lambda^{-\frac{1}{2}}\xi\right\|_{l^\infty(L^2)}$ and $\left\|\rho^{\frac{1}{2}}\chi^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\le\left\|\rho^{\frac{1}{2}}\chi\right\|_{l^\infty(L^2)}$, then

$$
\begin{aligned}
&\left\|\rho^{\frac{1}{2}}\chi^{n+1}\right\|_{L^2(\Omega)}^2+\left\|\lambda^{-\frac{1}{2}}\phi^{n+1}\right\|_{L^2(\Omega)}^2+\Delta t^2\left(\theta-\frac{1}{4}\right)\left\|\lambda^{-\frac{1}{2}}\xi^{n+1}\right\|_{L^2(\Omega)}^2\\
&\le 4\Delta t^2\left(\theta-\frac{1}{4}\right)\left\|\lambda^{-\frac{1}{2}}\xi\right\|_{l^\infty(L^2)}\left(\sum_{i=0}^{n}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\right)\\
&\quad+\frac{2\Delta t}{\rho_0^{\frac{1}{2}}}\left\|\rho^{\frac{1}{2}}\chi\right\|_{l^\infty(L^2)}\left(\sum_{i=0}^{n}\left\|\rho\bar{\partial}_t\eta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}+\sum_{i=0}^{n}\left\|\mathbf{R}^i\right\|_{L^2(\Omega)}\right)\\
&\quad+2\Delta t\left\|\lambda^{-\frac{1}{2}}\phi\right\|_{l^\infty(L^2)}\left(\sum_{i=0}^{n}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\right).
\end{aligned}
\tag{59}
$$

Applying the algebraic inequality: $ab\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$ to the right-hand side of (59) shows that

$$
\begin{aligned}
&\left\|\rho^{\frac{1}{2}}\chi^{n+1}\right\|_{L^2(\Omega)}^2+\left\|\lambda^{-\frac{1}{2}}\phi^{n+1}\right\|_{L^2(\Omega)}^2+\Delta t^2\left(\theta-\frac{1}{4}\right)\left\|\lambda^{-\frac{1}{2}}\xi^{n+1}\right\|_{L^2(\Omega)}^2\\
&\le \frac{1}{2}\Delta t^2\left(\theta-\frac{1}{4}\right)\left\|\lambda^{-\frac{1}{2}}\xi\right\|_{l^\infty(L^2)}^2+8\left(\theta-\frac{1}{4}\right)\left(\Delta t\sum_{i=0}^{N-1}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\right)^2\\
&\quad+\frac{1}{2}\left\|\rho^{\frac{1}{2}}\chi\right\|_{l^\infty(L^2)}^2+C\Delta t^2\left(\sum_{i=0}^{N-1}\left\|\rho\bar{\partial}_t\eta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}+\sum_{i=0}^{N-1}\left\|\mathbf{R}^i\right\|_{L^2(\Omega)}\right)^2\\
&\quad+\frac{1}{2}\left\|\lambda^{-\frac{1}{2}}\phi\right\|_{l^\infty(L^2)}^2+4\left(\Delta t\sum_{i=0}^{N-1}\left\|\lambda^{-\frac{1}{2}}\zeta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\right)^2.
\end{aligned}
\tag{60}
$$

If we take the supremum over n on the left-hand side and use the fact that $N\Delta t=T$, we conclude that

$$
\begin{aligned}
&\left\|\rho^{\frac{1}{2}}\chi\right\|_{l^\infty(L^2)}^2+\left\|\lambda^{-\frac{1}{2}}\phi\right\|_{l^\infty(L^2)}^2\le C\left\|\lambda^{-\frac{1}{2}}\zeta\right\|_{l^\infty(L^2)}^2\\
&\quad+C\Delta t^2\left(\sum_{i=0}^{N-1}\left\|\rho\bar{\partial}_t\eta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\right)^2+C\Delta t^2\left(\sum_{i=0}^{N-1}\left\|\mathbf{R}^i\right\|_{L^2(\Omega)}\right)^2.
\end{aligned}
\tag{61}
$$

For the case $\theta<\frac{1}{4}$, we can follow the analysis presented in [15,16] to derive error estimates similar to (61) under condition (50).

To complete the proof, we need to bound each term on the right-hand side of (61). The first term can be bounded using the approximation properties. Similarly, we have

$$
\Delta t\sum_{i=0}^{N-1}\left\|\rho\eta^{i+\frac{1}{2}}\right\|_{L^2(\Omega)}\le C\left(h^k\|\mathbf{u}\|_{L^\infty(H^k(\Omega))}+\Delta t^2\left\|\frac{\partial^3\mathbf{u}}{\partial t^3}\right\|_{L^1(0,T;L^2(\Omega))}\right).
$$

For the last term on the right-hand side of (61), we have

$$
\begin{aligned}
\Delta t\sum_{i=0}^{N-1}\|\mathbf{R}^i\|_{L^2(\Omega)}&\le C\|\mathbf{R}\|_{l^\infty(L^2)}\\
&\le C\Delta t\sum_{i=1}^{N-1}\|\mathbf{r}^i\|_{L^2(\Omega)}+C\|\rho(\Pi_h\mathbf{v}^0-\mathbf{v}^0)\|_{L^2(\Omega)}\\
&\quad+C\theta\Delta t^2\left\|\rho\bar{\partial}_t\mathbf{u}_{tt}^{\frac{1}{2}}\right\|_{L^2(\Omega)}+C\left\|\frac{1}{2\Delta t}\int_0^{\Delta t}\rho(\Delta t-t)^2\frac{\partial^3\mathbf{u}}{\partial t^3}(t)dt\right\|_{L^2(\Omega)}.
\end{aligned}
$$

To estimate $\|\mathbf{r}^i\|_{L^2(\Omega)}$, we make use of the identity

$$
\bar{\partial}_{tt}\mathbf{u}^n=u_{tt}^n+\frac{1}{6\Delta t^2}\int_{-\Delta t}^{\Delta t}(\Delta t-|s|)^3\frac{\partial^4\mathbf{u}}{\partial t^4}(t^n+s)ds.
\tag{62}
$$

From the Taylor's expansions of $\mathbf{u}_{tt}^{n+1}$ and $\mathbf{u}_{tt}^{n-1}$ about $\mathbf{u}_{tt}^{n}$;

$$\mathbf{u}_{tt}^{n+1} = \mathbf{u}_{tt}^{n} + \Delta t \mathbf{u}_{ttt}^{n} + \int_0^{\Delta t} (\Delta t - |s|) \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) ds,$$

and

$$\mathbf{u}_{tt}^{n-1} = \mathbf{u}_{tt}^{n} - \Delta t \mathbf{u}_{ttt}^{n} + \int_{-\Delta t}^{0} (\Delta t - |s|) \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) ds,$$

we obtain

$$\mathbf{u}_{tt}^{n;\theta} = \mathbf{u}_{tt}^{n} + \theta \int_{-\Delta t}^{\Delta t} (\Delta t - |s|) \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) ds. \tag{63}$$

Subtracting (62) from (63) yields

$$\mathbf{u}_{tt}^{n;\theta} - \overline{\partial}_{tt} \mathbf{u}^n = \frac{1}{6\Delta t^2} \int_{-\Delta t}^{\Delta t} (|s| - \Delta t)^3 \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) ds - \theta \int_{-\Delta t}^{\Delta t} (|s| - \Delta t) \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) ds.$$

Hence,

$$\| \mathbf{r}^i \|_{L^2(\Omega)}^2 = \| \rho \left(\mathbf{u}_{tt}^{n;\theta} - \overline{\partial}_{tt} \mathbf{u}^n \right) \|_{L^2} \le C \Delta t^3 \int_{-\Delta t}^{\Delta t} \left\| \rho^{\frac{1}{2}} \frac{\partial^4 \mathbf{u}}{\partial t^4}(t^n + s) \right\|_{L^2(\Omega)}^2 ds \le C \Delta t^4 \left\| \rho^{\frac{1}{2}} \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{L^\infty(L^2)}^2,$$

and therefore

$$\Delta t \sum_{i=1}^{n} \| \mathbf{r}^i \|_{L^2(\Omega)} \le C \Delta t^2 \left\| \rho^{\frac{1}{2}} \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{L^\infty(L^2)} \sum_{i=1}^{n} \Delta t \le C T \Delta t^2 \left\|^2 \rho^{\frac{1}{2}} \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{L^\infty(L^2)}.$$

Similarly, we have

$$\left\| \rho \overline{\partial}_t \mathbf{u}_{tt}^{\frac{1}{2}} \right\|_{L^2(\Omega)}^2 = \left\| \Delta t \int_0^{\Delta t} \rho \frac{\partial^3 \mathbf{u}}{\partial t^3}(t) dt \right\|_{L^2(\Omega)}^2 \le C \Delta t^3 \int_0^{\Delta t} \left\| \rho^{\frac{1}{2}} \frac{\partial^3 \mathbf{u}}{\partial t^3} \right\|_{L^2(\Omega)}^2 dt \le C \Delta t^4 \left\| \rho^{\frac{1}{2}} \frac{\partial^3 \mathbf{u}}{\partial t^3} \right\|_{L^\infty(L^2)}^2,$$

and

$$\left\| \frac{1}{2\Delta t} \int_0^{\Delta t} \rho (\Delta t - t)^2 \frac{\partial^3 \mathbf{u}}{\partial t^3}(t) dt \right\|_{L^2(\Omega)}^2 \le C \Delta t^3 \int_0^{\Delta t} \left\| \rho^{\frac{1}{2}} \frac{\partial^3 \mathbf{u}}{\partial t^3} \right\|_{L^2(\Omega)}^2 dt \le C \Delta t^4 \left\| \rho^{\frac{1}{2}} \frac{\partial^3 \mathbf{u}}{\partial t^3} \right\|_{L^\infty(L^2)}^2.$$

Finally, using the approximation property (20) and combining all the bounds, we arrive at

$$\Delta t \sum_{i=0}^{N-1} \| \mathbf{R}^i \|_{L^2(\Omega)} \le C(h^k + \Delta t^2),$$

which completes the proof of the desired estimate. □

Remarks. It is worthwhile to mention that the time discretization method is fourth-order accurate when $\theta = 1/12$. To preserve the temporal accuracy of the finite element scheme one has to modify (30) carefully to obtain an appropriate initial value $\mathbf{U}^1$. The analysis presented in [25] can be used to derive optimal a priori error estimates in this case.

7. Conclusions

We proposed and analyzed a family of fully discrete mixed finite element schemes for solving the acoustic wave equation. We derived stability conditions for conditionally implicit stable schemes covering the explicit case treated by Jenkins, Rivière and Wheeler [10]. The error estimates established provided optimal convergence rates for the use of mixed finite elements methods in solving the acoustic wave equation.

References

1. Baker, G.A. Error estimates for finite element methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 1976, **13,** 564-576.
2. Bao, H., Bielak, J., Ghattas, O., Kallivokas, L.F., O'Hallaron, D.R., Shewchuk, J.R. and Xu, J. Large-scale simulation of elastic wave propagation in heterogeneous media on parallel computers. *Comput. Methods Appl. Mech. Engrg.*, 1998, **152,** 85–102.
3. Cohen, G., Joly, P., Roberts, J.E. and Tordjman, N. Higher order triangular finite elements with mass lumping for the wave equation. *SIAM J. Numer. Anal.*, 2001, **38,** 2047-2078.
4. Dupont, T. L^2-estimates for Galerkin methods for second order hyperbolic equations. SIAM J. Numer. Anal., 1973, **10,** 880–889.
5. Marfurt, K.J. Accuracy of finite-difference and finite-element modeling of the scalar and elastic wave equations. *Geophysics*, 1984, **49,** 533-549.

6. Rauch, J. On convergence of the finite element method for the wave equation. *SIAM J. Numer. Anal.*, 1985, **22,** 245–249.
7. Cowsar, L.C., Dupont, T.F. and Wheeler, M.F. A priori estimates for mixed finite element methods for the wave equation. *Comput. Methods App. Mech. Engg.*, 1990, **82,** 205-222.
8. Cowsar, L.C., Dupont, T.F. and Wheeler, M.F. A priori estimates for mixed finite element approximations of second-order hyperbolic equations with absorbing boundary conditions. *SIAM J. Numer. Anal.*, 1996, **33,** 492-504.
9. Geveci, T. On the application of mixed finite methods to the wave equation. *RAIRO. Modél. Math. Anal. Numér.*,1988, **22,** 243–250.
10. Jenkins, E.W., Rivière, B. and Wheeler, M.F. A priori error estimates for mixed finite element approximations of the acoustic wave equation. *SIAM J. Numer. Anal.*, 2002, **40,** 1698–1715.
11. Pani, A.K. and Yuan, J.Y. Mixed finite element methods for a strongly damped wave equation. *Numer. Methods Partial Differential Equations,* 2001, **17(2),** 105-119.
12. Pani, A.K., Sinha, R.K. and Otta, A.K. An H^1-Galerkin mixed method for second order hyperbolic equations. *Int. J. Numer. Anal. Model.*, 2004, **1(2),** 111-130.
13. Vdovina, T. and Minkoff, S.E. An a priori error analysis of operator upscaling for the acoustic wave equation. *Inter. J. Numer. Anal. model.*, 2008, **5,** 543-569.
14. Grote, M.J. and Schötzau, D. Optimal error estimates for the fully discrete interior penalty DG method for the wave equation. *J. Sci. Comput.*, 2009, **40,** 257-272.
15. Johnson, C. Discontinuous Galerkin finite element methods for second order hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 1993, **107,** 117–129.
16. Rivière B. and Wheeler, M.F. Discontinuous finite element methods for acoustic and elastic wave problems. Part I: Semidiscrete error estimates. *TICAM report 01–02, University of Texas, Austin, TX, 2001.*
17. Rivière, B. and Wheeler, M.F. Discontinuous finite element methods for acoustic and elastic wave problems. *in ICM2002-Beijing Satellite Conference on Scientific Computing, Contemporary Mathematics, AMS, Providence*, 2003, **329**, 271-282.
18. Glowinski, R., Kinton, W. and Wheeler, M.F. A mixed finite element formulation for boundary controllability of the wave equation. *Int. J. Num. Meth. Engng.*, 1989, **27**, 623-635.
19. Knops, R.J. and Payne, L.E. Uniqueness Theorems in Linear Elasticity. *Springer Tracts Nat. Philos*, 19, *Springer-Verlag, Berlin*, 1971.
20. Jenkins, E.W. Numerical solution of the acoustic wave equation using Raviart-Thomas elements. *J. Comput. Appl. Math.,* 2007, **206,** 420–431.
21. Raviart, R.A. and Thomas, J.M. *Mathematical Aspects of the Finite Element Method*. Lecture Notes in Mathematics, Springer, Berlin, 1997, **106,** 292-315.
22. Nedelec, J.C. Mixed finite elements in $\mathbb{R}^n$**.** *Numer. Math.*, 1980, **35,** 315–341.
23. Brezzi, F., Douglas, Jr. J. and Marini, L.D. Two families of mixed elements for second order elliptic problems. *Numer. Math.,* 1985, **88,** 217–235.
24. Brezzi, F., Douglas, J., Fortin, Jr. M. and Marini, L.D. Efficient rectangular mixed finite elements in two and three space variables. *RAIRO Modèl. Math. Anal. Numér.,* 1987, **21,** 581-604.
25. Karaa, S. Finite element θ-schemes for the acoustic wave equation. *Adv. Appl. Math. Mech.,* 2011, **3,** 181-203.
26. Karaa, S. Stability and convergence of fully discrete finite element methods for the acoustic wave equation. *J. Appl. Math. Comput.,* 2012, **40,** 659-682.

Received 2 October 2014
Accepted 26 March 2015