SQU Journal for Science, 2015, 19(2), 55-61

A Unique Common Coupled Fixed Point Theorem for Four Maps in Partial b-Metric-Like Spaces

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ABSTRACT: We prove the existence of a unique common coupled fixed point theorem for four mappings satisfying a general contractive condition on partial b-metric-like spaces. We also give an example to illustrate our main theorem. Our theorem generalizes and improves the theorem of [1].

Keywords: b-Metric-like space; Coupled fixed point; w-Compatible maps.

نظرية النقطة الثابتة المزدوجة العامة لاربعة اقترانات في شبه الفراغ الجزئي ب

محمد سعيد خان ، كاندورو ب.ر.راو و كانديبالي ف.س. برافاثي

ملخص: في هذه الورقة قمنا بإثبات نظرية وجود نقطة ثابتة مزدوجة عامة لأربعة إقترانات تحقق شروط الانقباض العامة على شبه الفراغ الجزئي ب. و قد أحضرنا مثالا لتوضيح نظريتنا الرئيسية. نظريتنا هي تعميم و تحديث للنظرية المثبتة في [1].

كلمات مفتاحية: شبه الفراغ الجزئي ب ، نقطة ثابتة مزدوجة و الدوال المتوافقة توافق W.

1. Introduction and Preliminaries

The concept of b-metric space was introduced by Czerwik [2] as follows:

Definition 1.1 [2]: A b-metric on a non-empty set X is a function $d : X \times X \to [0, \infty)$ such that for all x, y, z ε X and a constant $k \geq 1$ the following three conditions hold true:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k\,[d(x, z) + d(z, y)]$.

The triad (X, d, k) is called a b-metric space.

Alghamdi *et al.* [3] introduced the concept of b-metric -like spaces and proved some fixed point theorems for a single map.

Definition 1.2 [3]: A b-metric-like on a non-empty set X is a function $d : X \times X \to [0, \infty)$ such that for all x, y, z ε X and a constant $k \geq 1$ the following three conditions hold true:

(i) $d(x, y) = 0$ implies $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The triad (X, d, k) is called a b-metric-like space.

Mathews [4] introduced the concept of a partial metric space as follows:

Definition 1.3 [4]: A mapping $p : X \times X \to [0, \infty)$, where X is a non-empty set, is said to be a partial metric on X if for any x, y, z ε X the following are satisfied:

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(ii) $p(x, x) \leq p(x, y)$,
(iii) $p(x, y) = p(y, x)$,

(iv) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

Now we give the following definition by combining the Definitions 1.2 and 1.3.

Definition 1.4: A partial b-metric-like on a non-empty set X is a function $p : X \times X \to [0, \infty)$ such that for all x, y, z ε X and a constant $k \ge 1$ the following are satisfied:

(p_1) $p(x, y) = 0$ implies $x = y$,
(p_2) $p(x, x) \le p(x, y)$, $p(y, y) \le p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \le k[p(x, z) + p(z, y) - p(z, z)]$.
The triad (X, p, k) is called a partial b-metric-like space.

Definition 1.5: Let (X, p, k) be a partial b-metric-like space and let $\{x_n\}$ be a sequence in X and x ε X. The sequence $\{x_n\}$ is said to be convergent to x if

$$\lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Definition 1.6: Let (X, p, k) be a partial b-metric-like space.

(i) A sequence $\{x_n\}$ in (X, p, k) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite .

(ii) A partial b-metric-like space (X, p, k) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point x ε X so that

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x) = \lim_{n\to\infty} p(x_n, x).$$

One can prove easily the following remark.

Remark 1.7: Let (X, p, k) be a partial b-metric-like space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then

(i) x is unique ,

(ii) $\frac{1}{k} p(x, y) \le \lim_{n\to\infty} p(x_n, y) \le k\, p(x, y)$ for all y ε X ,

(iii) $p(x_n, x_0) \le kp(x_0, x_1) + k^2p(x_1, x_2) + \cdots + k^{n-1}p(x_{n-2}, x_{n-1}) + k^n p(x_{n-1}, x_n)$ whenever $\{x_k\}_{k=0}^{n}$ ε X.

Let (X, p, k) be a partial b-metric-like space and $F, G : X \times X$ and $f, g : X \to X$. For x, y, u, v ε X, we denote

$$M_{u,v}^{x,y} = \min\left\{\begin{array}{c} p(fx, gu), p(fy, gv), p(fx, F(x, y)), p(fy, F(y, x)), \\ p(gu, G(u, v)), p(gv, G(v, u)), \\ \frac{1}{2k}[p(fx, G(u, v)) + p(gu, F(x, y))], \\ \frac{1}{2k}[p(fy, G(v, u)) + p(gv, F(y, x))] \end{array}\right\}.$$

and

$$m_{u,v}^{x,y} = \max\left\{\begin{array}{c} p(fx, F(x, y)), p(fy, F(y, x)), \\ p(gu, G(u, v)), p(gv, G(v, u)), \\ \frac{1}{k}p(fx, G(u, v)),\ \frac{1}{k} p(gu, F(x, y)), \\ \frac{1}{k}p(fy, G(v, u)),\ \frac{1}{k} p(gv, F(y, x)) \end{array}\right\}.$$

Recently Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point and discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems. Later Lakshmikantham and Ciric [6] proved some coupled coincidence and coupled common fixed point results in partially ordered metric spaces.

Definition 1.8 [6] An element (x, y) ε X × X is called
(i) a coupled coincident point of mappings F : X×X → X and g: X → X if gx = F (x, y) and gy = F (y, x).
(ii) a common coupled fixed point of mappings F : X × X → X and g : X → X
if x = gx = F (x, y) and y = gy = F (y, x) .

Definition 1.9 [7] The mappings F : X × X → X and g : X → X are called w-compatible if g(F (x, y)) = F (gx, gy) and g(F (y, x)) = F (gy, gx), whenever
gx = F (x, y) and gy = F (y, x).

Recently, Abbas *et al*. [8] proved a common fixed point theorem for two maps of Jungck type satisfying generalized condition (B) in metric spaces (See Theorem 2.2, [8]). As a generalization of Theorem 2.2 of [8], Kaewcharoen *et al*. [1] obtained a common fixed point theorem for four maps satisfying a generalized condition in partial metric spaces.
In this paper, we obtain the existence of a unique common coupled fixed point theorem for four mappings satisfying a general contractive condition on partial b-metric-like spaces. We also give an example to illustrate our main theorem. Our theorem generalizes and improves the theorems of [1] and [8].

2. Main Result

Theorem 2.1: Let (X, p, k) be a complete partial b-metric-like space, F, G : X × X → X and f, g : X → X be mappings satisfying
(2.1.1) F (X × X) $\subseteq$ g(X), G(X × X) $\subseteq$ f(X),
(2.1.2) k p(F(x,y),G(u,v)) ≤ δ $M_{u,v}^{x,y}$ + L $m_{u,v}^{x,y}$
for all x, y, u, v ε X, where δ > 0 and L ≥ 0, k l < 1, where l = max $\left\{\frac{L}{1-\delta}, \delta + L\right\}$,
(2.1.3) f(X) or g(X) is closed,
(2.1.4) the pairs (F, f), and (G, g) are w-compatible.
Then F, G, f and g have a unique common coupled fixed point.

Proof. Let (x_0, y_0) ε X × X. Since F (X × X) $\subseteq$ g(X), there exist x_1, y_1 ε X such that gx_1 = F (x_0, y_0) and gy_1 = F (y_0, x_0). Since G (X × X) $\subseteq$ f(X), there exist x_2, y_2 ε X such that fx_2 = G(x_1, y_1) and fy_2 = G(y_1, x_1). Continuing this process, we construct sequences {x_n} and {y_n} in X such that

$$gx_{2n+1} = F(x_{2n}, y_{2n}) = z_{2n},$$
$$gy_{2n+1} = F(y_{2n}, x_{2n}) = w_{2n},$$
$$fx_{2n+2} = G(x_{2n+1}, y_{2n+1}) = z_{2n+1},$$
$$fy_{2n+2} = F(y_{2n+1}, x_{2n+1}) = w_{2n+1},\ n = 0, 1, 2, 3, \cdots$$

Now consider

$$p(z_{2n}, z_{2n+1}) \le k\, p(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}))$$
$$\le \delta\, M_{x_{2n+1},y_{2n+1}}^{x_{2n},y_{2n}} + L m_{x_{2n+1},y_{2n+1}}^{x_{2n},y_{2n}} \quad (1)$$

where

$$M_{x_{2n+1},y_{2n+1}}^{x_{2n},y_{2n}} = \max\left\{\begin{array}{l} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), p(z_{2n-1}, z_{2n}), \\ p(w_{2n-1}, w_{2n}), p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1}), \\ \frac{1}{2k}[p(z_{2n-1}, z_{2n+1}) + p(z_{2n}, z_{2n})], \\ \frac{1}{2k}[p(w_{2n-1}, w_{2n+1}) + p(w_{2n}, w_{2n})] \end{array}\right\}$$

$$\le \max\left\{\begin{array}{l} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1}) \end{array}\right\} \text{ from } k \ge 1 \text{ and from } (p_4)$$

$$m^{x_{2n},y_{2n}}_{x_{2n+1},y_{2n+1}} = \min\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}), \\ \frac{1}{k} p(z_{2n-1}, z_{2n+1}),\ \frac{1}{k}\ p(z_{2n}, z_{2n}), \\ \frac{1}{k} p(w_{2n-1}, w_{2n+1}),\ \frac{1}{k}\ p(w_{2n}, w_{2n}) \end{array}\right\}$$

$$\leq \min\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), p(z_{2n},z_{2n+1}), \\ p(w_{2n},w_{2n+1}), p(z_{2n-1}, z_{2n}) + p(z_{2n}, z_{2n+1}), \\ p(z_{2n}, z_{2n}), p(w_{2n-1}, w_{2n}) + p(w_{2n}, w_{2n+1}), p(w_{2n}, w_{2n}) \end{array}\right\}$$

$= \min\{p(z_{2n}, z_{2n}), p(w_{2n}, w_{2n})\}$, from (p_2)
$\leq \max\{p(z_{2n}, z_{2n}), p(w_{2n}, w_{2n})\}$
$\leq \max\{p(z_{2n}, z_{2n-1}), p(w_{2n}, w_{2n-1})\}$, from (p_2).

Thus

$$p(z_{2n}, z_{2n+1}) \leq \delta \max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}) \end{array}\right\} + L\max\{p(z_{2n}, z_{2n-1}), p(w_{2n}, w_{2n-1})\}.$$

Similarly

$$p(w_{2n},w_{2n+1}) \leq \delta \max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}) \end{array}\right\} + L\max\{p(z_{2n}, z_{2n-1}), p(w_{2n}, w_{2n-1})\}.$$

Thus

$$\max\left\{\begin{array}{c} p(z_{2n}, z_{2n+1}), \\ p(w_{2n},w_{2n+1}) \end{array}\right\} \leq \delta \max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}) \end{array}\right\} + L\max\left\{\begin{array}{c} p(z_{2n}, z_{2n-1}), \\ p(w_{2n},w_{2n-1}) \end{array}\right\} \quad (2)$$

If $\max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}) \end{array}\right\} \leq \max\left\{\begin{array}{c} p(z_{2n}, z_{2n+1}), \\ p(w_{2n},w_{2n+1}) \end{array}\right\}$,

then from (2)

$$\max\{p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1})\} \leq \frac{L}{1-\delta} \max\{p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n})\}.$$

If $\max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n}), \\ p(z_{2n},z_{2n+1}), p(w_{2n},w_{2n+1}) \end{array}\right\} \leq \max\left\{\begin{array}{c} p(z_{2n-1}, z_{2n}), \\ p(w_{2n-1},w_{2n}) \end{array}\right\}$,

then from (2)

$$\max\{p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1})\} \leq (\delta + L)\max\{p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n})\}.$$

Hence

$$\max\{p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1})\} \leq l\max\{p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n})\}$$

where $l = \max\left\{\frac{L}{1-\delta}, \delta + L\right\} < 1$.

Similarly we can show that

$$\max\{p(z_{2n-1}, z_{2n}), p(w_{2n-1}, w_{2n})\} \leq l\max\{p(z_{2n-2}, z_{2n-1}), p(w_{2n-2}, w_{2n-1})\}.$$

Hence

$$\text{Max}\{p(z_n, z_{n+1}), p(w_n, w_{n+1})\} \leq l\max\{p(z_{n-1}, z_n), p(w_{n-1}, w_n)\}, \quad n = 1, 2, 3, \cdots$$

Thus

$$\max\{p(z_n, z_{n+1}), p(w_n, w_{n+1})\} \leq l^n \max\{p(z_0, z_1), p(w_0, w_1)\}. \quad (3)$$

From (3), it follows that

$$\lim_{n\to\infty} p(z_n, z_{n+1}) = 0 = \lim_{n\to\infty} p(w_n, w_{n+1}). \quad (4)$$

For m > n, consider
$\max\{p(z_n, z_m), p(w_n, w_m)\}$

$$\leq \max\left\{\begin{array}{c} k\,p(z_n,z_{n+1}) + k^2\ p(z_{n+1},z_{n+2}) + \ldots + k^{m-n-1}\ p(z_{m-2},z_{m-1}) + k^{m-n-1}\ p(z_{m-1},z_m), \\ k\,p(w_n,w_{n+1}) + k^2\ p(w_{n+1},w_{n+2}) + \ldots + k^{m-n-1}\ p(w_{m-2},w_{m-1}) + k^{m-n-1}\ p(w_{m-1},w_m) \end{array}\right\}$$

$\leq k\max\{p(z_n, z_{n+1}), p(w_n, w_{n+1})\} + k^2\max\{p(z_{n+1}, z_{n+2}), p(w_{n+1}, w_{n+2})\}$
$+ \ldots + k^{m-n-1}\max\{p(z_{m-2}, z_{m-1}), p(w_{m-2}, w_{m-1})\} +$
$k^{m-n-1}\max\{p(z_{m-1}, z_m), p(w_{m-1}, w_m)\}$

$\leq \left(kl^n + k^2 l^{n+1} + ... + k^{m-n-1} l^{m-2} + k^{m-n-1} l^{m-1}\right) \max \left\{ \begin{matrix} p(z_0, z_1), \\ p(w_0, w_1) \end{matrix} \right\}$

$= kl^n \left(1 + kl^1 + k^2 l^2 + ... + k^{m-n-2} l^{m-n-2} + k^{m-n-2} l^{m-n-1}\right) \max \left\{ \begin{matrix} p(z_0, z_1), \\ p(w_0, w_1) \end{matrix} \right\} \leq$

$kl^n \left(1 + kl^1 + k^2 l^2 + ... + k^{m-n-2} l^{m-n-2} + k^{m-n-1} l^{m-n-1}\right) \max \left\{ \begin{matrix} p(z_0, z_1), \\ p(w_0, w_1) \end{matrix} \right\}$

$\leq \dfrac{kl^n}{1-kl} \max \left\{ \begin{matrix} p(z_0, z_1), \\ p(w_0, w_1) \end{matrix} \right\}$, since $kl < 1$.

Hence

$$\lim_{n,m\to\infty} p(z_n, z_m) = 0 = \lim_{n,m\to\infty} p(w_n, w_m). \tag{5}$$

Thus $\{z_n\}$ and $\{w_n\}$ are Cauchy in (X, p, k).

Since X is complete, the sequences $\{z_n\}$ and $\{w_n\}$ converge to some α and β in X respectively such that

$\lim_{n,m\to\infty} p(z_n, z_m) = p(\alpha, \alpha)$ and $\lim_{n,m\to\infty} p(w_n, w_m) = p(\beta, \beta)$.

Also $\lim_{n\to\infty} p(z_n, \alpha) = p(\alpha, \alpha)$ and $\lim_{n\to\infty} p(w_n, \beta) = p(\beta, \beta)$.

Now from (5), we have

$$p(\alpha, \alpha) = 0 = p(\beta, \beta). \tag{6}$$

Suppose f(X) is closed.

Since $fx_{2n+2} = z_{2n+1} \to \alpha$ and $fy_{2n+2} = w_{2n+1} \to \beta$, it follows that α = fu and β = fv for some u, v ε X.

Consider

$$\begin{aligned} p(\alpha, F(u, v)) &\leq kp(\alpha, z_{2n+1}) + kp(F(u, v), G(x_{2n+1}, y_{2n+1})) \\ &\leq kp(\alpha, z_{2n+1}) + \delta\, M^{u,v}_{x_{2n+1}, y_{2n+1}} + L\, m^{u,v}_{x_{2n+1}, y_{2n+1}} \end{aligned}$$

$$M^{u,v}_{x_{2n+1}, y_{2n+1}} = \max \left\{ \begin{matrix} p(fu, z_{2n}), p(fv, w_{2n}), p(fu, F(u, v)), \\ p(fv, F(v, u)), p(z_{2n}, z_{2n+1}), p(w_{2n}, w_{2n+1}), \\ \frac{1}{2k}[p(fu, z_{2n+1}) + p(z_{2n}, F(u, v))], \\ \frac{1}{2k}[p(fv, w_{2n+1}) + p(w_{2n}, F(v, u)] \end{matrix} \right\}$$

$$\to \max \left\{ \begin{matrix} 0, 0, p(\alpha, F(u, v)), \\ p(\beta, F(v, u)), 0, 0, \\ \frac{1}{2k}[0 + p(\alpha, F(u, v))], \\ \frac{1}{2k}[0 + p(\beta, F(v, u)] \end{matrix} \right\}, \text{ from (4) and Remark 1.7 (ii)}$$

$= \max \{ p(\alpha, F(u, v)), p(\beta, F(v, u))\}$.

Also $m^{u,v}_{x_{2n+1}, y_{2n+1}} \to 0$.

Thus

$$p(\alpha, F(u, v)) \leq \delta \max \{p(\alpha, F(u, v)), p(\beta, F(v, u))\} .$$

Similarly we can show that

$$p(\beta, F(u, v)) \leq \delta \max \{p(\alpha, F(u, v)), p(\beta, F(v, u))\} .$$

Hence

$$\max \{p(\alpha, F(u, v)), p(\beta, F(v, u))\} \leq \delta \max \{p(\alpha, F(u, v)), p(\beta, F(v, u))\},$$

which in turn yields that α = F (u, v) and β = F (v, u).

Thus fu = α = F (u, v) and fv = β = F (v, u).

Since the pair (F, f) is w- compatible, we have

$$f\alpha = F(\alpha, \beta) \text{ and } f\beta = F(\beta, \alpha). \tag{7}$$

Since α = F (u, v) ε F (X × X) ⊆ g(X), there exists r ε X such that α = gr.

Since β = F (v, u) ε F (X × X) ⊆ g(X), there exists t ε X such that β = gt.

Now $p(\alpha, G(r,t)) \leq s\, p(F(u,v), G(r,t)) \leq \delta\, M_{r,t}^{u,v} + L\, m_{r,t}^{u,v}$

$$M_{r,t}^{u,v} = \max\left\{\begin{array}{c} p(fu, gr), p(fv, gt), p(fu, F(u,v)), \\ p(fv, F(v,u)), p(gr, G(r,t)), p(gt, G(t,r)), \\ \frac{1}{2k}[p(fu, G(r,t)) + p(gr, F(u,v))], \\ \frac{1}{2k}[p(fv, G(t,r)) + p(gt, F(v,u)] \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} 0, 0, 0, 0 \\ p(\alpha, G(r,t)), p(\beta, G(t,r)), \\ \frac{1}{2k}[p(\alpha, G(r,t)) + p(\beta, F(u,v))], \\ \frac{1}{2k}[p(\beta, G(t,r)) + p(\beta, F(v,u)] \end{array}\right\}$$

$= \max\{ p(\alpha, G(r,t)), p(\beta, G(t,r))\}$.

$m_{r,t}^{u,v} = 0$.

Thus

$$p(\alpha, G(r,t)) \leq \delta \max\{p(\alpha, G(r,t)), p(\beta, G(t,r))\}.$$

Similarly we can show that

$$p(\beta, G(t,r)) \leq \delta \max\{p(\alpha, G(r,t)), p(\beta, G(t,r))\}.$$

Hence

$$\max\{p(\alpha, G(r,t)), p(\beta, G(t,r))\} \leq \delta \max\{p(\alpha, G(r,t)), p(\beta, G(t,r))\}$$

which in turn yields that $gr = \alpha = G(r,t)$ and $gt = \beta = G(t,r)$.

Since the pair (G, g) is w-compatible, we have

$$g\alpha = G(\alpha, \beta) \text{ and } g\beta = G(\beta, \alpha).$$

Now consider

$$p(f\alpha, \alpha) \leq k\, p(F(\alpha, \beta), G(r,t)) \leq \delta\, M_{r,t}^{\alpha,\beta} + L\, m_{r,t}^{\alpha,\beta}$$

$$M_{r,t}^{\alpha,\beta} = \max\left\{\begin{array}{c} p(f\alpha, gr), p(f\beta, gt), p(f\alpha, F(\alpha,\beta)), \\ p(f\beta, F(\beta,\alpha)), p(gr, G(r,t)), p(gt, G(t,r)), \\ \frac{1}{2k}[p(f\alpha, G(r,t)) + p(gr, F(\alpha,\beta))], \\ \frac{1}{2k}[p(f\beta, G(t,r)) + p(gt, F(\beta,\alpha))] \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} p(f\alpha, \alpha), p(f\beta, \beta), 0, 0, 0, 0, \\ \frac{1}{2k}[p(f\alpha, \alpha) + p(\alpha, f\alpha)], \\ \frac{1}{2k}[p(f\beta, \beta) + p(\beta, f\beta)] \end{array}\right\}.$$

$= \max\{ p(f\alpha, \alpha), p(f\beta, \beta)\}$.

Also $m_{r,t}^{\alpha,\beta} = 0$.

Thus $p(f\alpha,\alpha) \leq \delta \max\{p(f\alpha, \alpha), p(f\beta, \beta)\}$.
Similarly we can show that

$$p(f\beta, \beta) \leq \delta \max\{p(f\alpha, \alpha), p(f\beta, \beta)\}.$$

Hence

$$\max\{p(f\alpha, \alpha), p(f\beta, \beta)\} \leq \delta \max\{p(f\alpha, \alpha), p(f\beta, \beta)\}$$

which in turn yields that $f\alpha = \alpha$ and $f\beta = \beta$.
Similarly we can show that $g\alpha = \alpha$ and $g\beta = \beta$.

Thus

F (α, β) = fα = α = gα = G(α, β) and
F (β, α) = fβ = β = gβ = G(β, α).
Hence (α, β) is a common coupled fixed point of F, G, f and g. Uniqueness of this common coupled fixed point follows easily from (2.1.2).

Now, we give an example to illustrate our main Theorem 2.1.

Example 2.2 Let X = [0, 1] and p(x, y) = max{x^2, y^2}. Then (X, p, k) is a complete partial b-metric-like space with k = 2. Define F, G : X × X → X and f, g : X → X as F(x, y) = 0, G(x, y) = $\frac{x}{4}$, fx = $\frac{x}{2}$ and gx = x. Then

$$k\, p(F(x, y), G(u, v)) = 2 \max\left\{0, \frac{u^2}{16}\right\} = \frac{u^2}{8},$$

$$p(gu, G(u, v)) = \max\left\{u^2, \frac{u^2}{16}\right\} = u^2.$$

Thus $p(F(x, y), G(u, v)) = \frac{1}{8} u^2 = \frac{1}{8} p(gu, G(u, v))$

$$\leq \frac{1}{8} M^{x,y}_{u,v} + 0\, m^{x,y}_{u,v}.$$

Here $\delta = \frac{1}{8}$, L = 0, $k\,l = \frac{1}{4} < 1$. Clearly (2.1.1), (2.1.3) and (2.1.4) are satisfied and (0, 0) is the unique common coupled fixed point of F, G, f and g.

Theorem 2.1 is a generalization and improvement of the following:

Theorem 2.3 (Theorem 2.1, [1]): Let (X, p) be a complete partial metric space. Suppose that f, g, F, G : X → X satisfying the following conditions
(2.3.1) f(X) ⊆ g(X) and F (X) ⊆ G(X),
(2.3.2) there exist δ > 0 and L ≥ 0 with δ + 2L < 1 such that
p(Fx, fy) ≤ δ M(x, y) + L min{p(gx, Fx), p(Gy, fy), p(gx, fy), Gy, Fx)}
for all x, y ε X, where
M(x, y) = max{p(gx, Gy), p(gx, Fx), p(Gy, fy), $\frac{1}{2}$ [p(gx, fy) + p(Gy, Fx)]},
(2.3.3)f(X) or g(X) is closed and
(2.3.4) the pairs (f, G) and (g, F) are w-compatible.
Then f, g, F and G have a unique common fixed point in X.

Acknowledgement

The authors are thankful to the referees for their valuable suggestions in improving the manuscript.

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Received 24 August 2014
Accepted 20 December 2014