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Incidence Matrices of Directed Graphs of Groups and their up-down Pregroups

Wadhah S. Jassim

Department of Mathematics, Faculty of Science, Soran University, Soran, Erbil, Iraq.
Email: wadhahjasim@maths.soran.edu.krd.

ABSTRACT: The aim of this work is to give a definition of the incidence matrices of the directed graph of groups, construct an up-down pregroup of the incidence matrices of the directed graph of groups and then give an algorithm for the up-down pregroup of the directed graph of groups.

Keywords: Incidence matrix of X-labeled graph; up-down pregroup; directed graph of groups and incidence matrix of a directed graph of groups.

مصفوفات الوقوع لبيانات الزمر الموجه و أعلى - أسفل ما قبل زمرها

وضاح س. جاسم

الملخص: هدف بحثنا هذا هو اعطاء تعريف لمصفوفات الوقوع لبيانات الزمرالموجه وأعلى – أسفل ما قبل زمرها ، بناء أعلى أسفل ما قبل زمره لمصفوفات الوقوع لبيانات الزمر الموجه ومن ثم اعطاء خوارزمية لبناء أعلى – أسفل ما قبل زمره لبيانات الزمر الموجه.

الكلمات المفتاحية: مصفوفة الوقوع للبيان المحمول بعناصر ا لمجموعه – X ، أعلى - أسفل ما قبل زمره ، بيانات الزمر الموجه ومصفوفة الوقوع لبيان الزمر الموجه.

1. Introduction

In [1] we gave the definition of the incidence Matrices of *X*- Labeled graphs. In [2], [3] we gave the definition of the directed graph of groups, constructed graph of groups for pregroups directly from the ordered tree of pregroups, and from that directed graph of groups we constructed the up-down pregroups, and then we showed those two pregroups are isomorphic. In [4] Rimlinger gave an example of a pregroup P of finite height; he said "but Jim Shearer and I spent a very long evening with the computer and verified the pregroup axioms". I bear this point in mind. In [2], [3] we have a direct method to obtain examples of pregroups in the form of up-down pregroups from any directed graph of groups, but sometimes those graphs of groups are large, and then will take a long time to find those up-down pregroups. In [1] we defined the incidence matrices of X-labeled graphs. The main aim of this work is to represent the directed graph of finite groups in terms of the incidence matrices of X-labeled graphs, so that by adding certain conditions to allow the incidence matrices of the X-labeled graph to be more confident with the definition of the directed graph of groups; we can then write a computer program to record all elements of the up-down pregroup of that directed graph of groups, as an application of the incidence matrices of X-labeled graph. Therefore, this paper is divided into six sections. In section 2, we give the basic concepts of graphs, pregroups and incidence matrices of X-labeled graphs. In section 3, we give the definition of incidence matrices of directed graphs of groups. In section 4, we construct the up-down pregroup of the incidence matrices of the directed graph of groups. In section 5, we define an algorithm on the incidence matrices of the directed graph of groups, so we can then write a computer program for this algorithm.

2. Basic concepts

2.1 Pregroups

The idea of pregroups goes back to Baer [5] and the definition of pregroup was given independently by Stallings [6] in 1971. The theory of pregroups has been developed by [4], Stallings [6], Hoare [7] and Hoare – Jassim [3] and others. We now return to the original definition of pregroups [6].

Let P be **a set with an element 1** $\in$ ***P*** and a mapping of a subset D of $P \times P$ into P, denoted by $(x, y) \mapsto xy$. We shall say that xy is defined instead of $(x, y) \in D$. Suppose that there is an involution on P denoted by $x \mapsto x^{-1}$, such that the following axioms hold:

P1: $x1 = 1x$ for all $x \in P$,

P2: $xx^{-1} = 1 = x^{-1}x$ for all $x \in P$,

P3: If xy is defined, then $y^{-1}x^{-1}$ is defined and $(xy)^{-1} = y^{-1}x^{-1}$.

P4: if xy and yz are defined then $(xy)z$ is defined if and only if $x(yz)$ is defined, in which case the two are equal and we will say xyz is defined.

P5: For any w, x, y and z in P, if wx, xy and yz are defined, then either wxy or xyz is defined .

Hoare **[7]** showed that we could prove axiom P3 above by using the following proposition, and axioms P1, P2 and P4.

***Definition* 2.2.** [7]**:** For any $x \in P$, put $L(x) = \{a \in P: ax$ is defined$\}$. We write $x \le y$ if $L(y) \subseteq L(x)$, $x < y$ if $L(y) \subset L(x)$ and $L(x) \ne L(y)$, and $x \sim y$ if $L(x) \ne L(y)$. It is clear that $\sim$ is an equivalence relation compatible with $\le$.

The following results are taken from Stallings [6] and Rimlinger [4]. (See [7] for shorter proofs).

***Proposition* 2.3.**

(i) If $x \le y$ or $y \le x$, then $x^{-1}y$ and $y^{-1}x$ are defined.

(ii) If xa and $a^{-1}y$ are defined, then $(xa)(a^{-1}y)$ is defined if and only if xy is defined, in which case they are equal.

By using axiom P5 above (which will be denoted by P5(i)) Rimlinger [4] proved conditions P5(ii) and P5(iii) of Lemma 2.4 below.

***Lemma* 2.4** [7]**.** The following conditions on elements of P are equivalent:

P5(i). If wx,xy and yz are defined , then either wxy or xyz is defined .

P5(ii). If $x^{-1}a$ and $a^{-1}y$ are defined but $x^{-1}y$ is not , then $a < x$ and $a < y$.

P5(iii). If $x^{-1}y$ is defined, then $x \le y$ or $y \le x$.

Therefore, we will say P is a pregroup if it satisfies axioms P1, P2, P4, and the conditions of Lemma 2.4, above. The universal group of a pregroup P **[**13**]** is denoted by $U(P)$ and has the following presentation $< P; x.y = xy$ whenever xy is defined, for $x, y, \in P >$. Now if P is a pregroup, then $(P, \le)$ is tree - like partial ordering; that is P/~ has a minimum element and, for any x,y and z in P , $x \le z$ and $y \le z$ we have $x \le y$ or $y \le x$. Moreover Rimlinger in **[4]** defined that for any element x in P, we say that x has ***finite height*** $n \ge 0$, if there exists a maximal totally ordered subset $\{x_0, x_1, \cdots, x_n\}$ of P such that $1 = x_0 < x_1 < \cdots < x_n = x$. He also showed that the elements of P form an **order tree** (denoted by ***O***) whose vertices , $[x]$, are the equivalence classes of the elements of P under ~, and whose edges e, are formed by joining each vertex $[x]$ of height $n > 0$ to the unique vertex $[y]$ of height $n - 1$ satisfying $[y] < [x]$, and all edges e of ***O*** are directed away the base vertex [x_0] of height 0. In **[8]** Stallings constructed an up – down pregroup for a free group F generated by $X = \{a, b\}$ of infinite height, and he showed that $U(P)$ the universal group of a pregroup P is isomorphic to F. In **[**2,3**]** we gave the definition of a **directed graph of groups** which consists of a directed graph Y, with a base vertex v^* and a spanning tree T, whose edges are directed away from the base vertex v^*, together with a group G_v for each vertex v and for each directed edge $e \in Y$, a subgroup G_e of $G_{i(e)}$ which is embedded in $G_{\tau(e)}$ by ψ_e which is defined by $\psi_e(a) = y_e^{-1} a y_e$, where $a \in G_e$ and y_e is the labeled of the edge e. It is denoted by $(G_v, G_e, Y, T, v^*, \psi_e)$. We also constructed a directed graph of groups of P directly from the order tree ***O*** of P and then showed that the fundamental group of a graph of groups $\pi_1(G_v, G_e, Y, T, v^*, \psi_e)$ is isomorphic to $U(P)$, We constructed an up – down pregroup Q directly from the directed graph of groups $(G_v, G_e, Y, T, v^*, \psi_e)$ of a pregroup P and we showed that $U(Q)$ is isomorphic to $\pi_1(G_v, G_e, Y, T, v^*, \psi_e)$ and then that $U(Q) \cong U(P)$.

2.5 Incidence Matrices of *X* – Labeled Graphs

In [1] we gave the definition of the incidence matrices of *X* – Labeled graphs (where an *X*-labeled graph is a **directed graph with each edge labeled by an element *x* of the subset *X*** of the group F and *X* generating the group *F*), and some definitions and results related to it. Recall that from graph theory the directed graphs Γ are without loops, because we cannot define the incidence matrices of directed graphs Γ. The incidence matrices of directed graphs Γ are with *n* vertices and *m* edges (i.e. it is $n \times m$ matrices $[x_{ij}]$, where $1 \le i \le n, 1 \le j \le m$) such that:

$$x_{ij} = \begin{cases} 1 & if \quad v_i = i(e_j) \\ 0 & if \quad v_i \; is \quad not incedence \quad with \quad e_j \\ -1 & if \quad v_i = \tau(e_j) \end{cases}$$

Since all edges *e* in *X* – Labeled graphs are labeled $x \in X \cup X^{-1}$ and the incidence matrices of the directed graphs **do not deal with the labeling of edges**, we will put more conditions on the incidence matrices of directed graphs as below to obtain the definition of the incidence matrices of the X- Labeled graphs.

Definition 2.6: Let Γ be any *X* – Labeled graph without loops (where $X = \{a,b\}$), then the **incidence matrix** of the *X* – Labeled graph Γ is an $n \times m$ incidence matrix $[x_{ij}]$, where $1 \le i \le n, 1 \le j \le m$) with x_{ij} entries such that

$$x_{ij} = \begin{cases} x \;\; if \; v_i = i(e_j) \quad and \quad e_j \quad labels \quad x \in X \\ 0 \; if \; v_i \quad is \quad not\, incident \quad with \quad e_j \\ x^{-1} \; if \; v_i = \tau(e_j) \quad and \quad e_j \quad labels \quad x \in X \end{cases}$$

N.B. Incidence matrices of *X* – Labeled graphs Γ will be denoted by $M_X(\Gamma)$. **For example:** the Cayley graph $\Gamma(F,X)$ of the group F generated by $X \subseteq F$, the Cayley coset graph $\Gamma(H)$ of the subgroup *H* of *F*, the core graph of the Cayley coset graph $\Gamma^*(H)$ of the subgroup *H* of *F* and the product of core graphs $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ are X-labeled graphs.

Now if $X = \{a,b\}$ and the *X* – Labeled graph Γ has **loops** with labeling *a* or *b*, then choose a mid point on all edges labeled *a* or *b* to make all of them two edges labeled *aa* or *bb* respectively . Therefore in the rest of this work we will **assume** that all *X* – Labeled graphs Γ are **without loops**.

Definition 2.7: Let $M_X(\Gamma)$ be an incidence matrix of *X* –Labeled graph Γ**.** If $M_X(\Gamma)$ doesn't contain any row r_i with non zero entries x_{ij} and x_{ik} in $X \cup X^{-1}$ such that $x_{ij} = x_{ik}$, then $M_X(\Gamma)$ is called a **folded incidence matrix** of *X* – Labeled graph Γ**.**

Now we give the **basic definitions and some results** on the incidence matrix of *X* – Labeled graph $M_X(\Gamma)$, as given in [1].

Let $M_X(\Gamma)$ be an $n \times m$ incidence matrix $[x_{ij}]$ of *X* – Labeled graphs Γ, and let r_i and c_j be a row and a column in $M_X(\Gamma)$ respectively. If x_{ij} is a non – zero entry in the row r_i, then r_i is called an **incidence row** with the column c_j at the non – zero entry $x_{ij} \in X \cup X^{-1}$, and if the non – zero entry $x_{ij} \in X$, then the row r_i is called the **starting row** (denoted by $s(c_j))$ of the column c_j, and the row r_i is called the **ending row** (denoted by $e(c_j)$) of the column c_j if $x_{ij} \in X^{-1}$. If the rows r_i and r_k are incident with column c_j at the non – zero entries x_{ij} and x_{kj} respectively, then we say that the rows r_i and r_k are **adjacent**. If c_j and c_h are two distinct columns in $M_X(\Gamma)$ such that the row r_i is incidence with the columns c_j and c_h at the non – zero entries x_{ij} and x_{ih} respectively (where $x_{ij}, x_h \in X \cup X^{-1}$), then we say that c_j and c_h are **adjacent columns**. For each column *c* there is an inverse column denoted by $\bar{c}$ such that $s(\bar{c}) = e(c), e(\bar{c}) = s(c)$ and $\bar{\bar{c}} = c$. The degree of a row r_i of

$M_X(\Gamma)$ is the number of the columns incident to r_i and is denoted by $\deg(r_i)$. If the row r_i is incident with at least three distinct columns c_j, c_h and c_k at the non – zero entries , then the row r_i is called a **branch row**. If the row r_i is incident with only one column c_j at the non- zero entry $x_{ij} \in X \cup X^{-1}$ and all other entries of r_i are zero, then the row r_i is called an **isolated row**. A **scale** in $M_X(\Gamma)$ is a finite sequence of form $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$, where $k \geq 1$, $\in = \mp$, $s(c_j^{\in_j}) = r_j$, and $e(c_j^{\in_j}) = r_{j+1} = s(c_{j+1}), 1 \leq j \leq k.$ .The starting row of a scale $S = r_1, c_1^{\in_1}, r_2, c_2^{\in_2}, \ldots, r_{k-1}, c_{k-1}^{\in_{k-1}}, r_k$ is the starting row r_1 of the column c_1 and the **ending row** of the scale *S* is the ending row r_k of the column c_{k-1} and we say that *S* is a scale from r_1 to r_k and *S* is a scale of length *k* for $1 \leq j \leq k-2$. If $s(S) = e(S)$, then the scale is called a **closed scale**. If the scale S is reduced and closed, then S is called a **circuit** or a **cycle.** If $M_X(\Gamma)$ has no cycle, then $M_X(\Gamma)$ is called a forest incidence matrix of *X* – Labeled graph Γ. Two rows r_i and r_k in $M_X(\Gamma)$ are called **connected** if there is a scale *S* in $M_X(\Gamma)$ containing r_i and r_k. Moreover $M_X(\Gamma)$ is called **connected** if any two rows r_i and r_k in $M_X(\Gamma)$ are connected by a scale *S*. If $M_X(\Gamma)$ is a connected and forest , then $M_X(\Gamma)$ is called a **tree** incidence matrix of *X* – Labeled graph Γ. Let Ω be a subgraph of Γ, then $M_X(\Omega)$ is called a **subincidence matrix** of $M_X(\Gamma)$, if the set of rows and columns of $M_X(\Omega)$ are subsets of $M_X(\Gamma)$ and if c is a column in $M_X(\Delta)$, then $s(c), e(c)$ and $\bar{c}$ have the same meaning in $M_X(\Gamma)$ as they do in $M_X(\Omega)$. If $M_X(\Omega) \neq M_X(\Gamma)$, then $M_X(\Omega)$ is called a **proper** subincidence matrix of $M_X(\Gamma)$. A **component** of $M_X(\Gamma)$ is a maximal connected **subincidence matrix** of $M_X(\Gamma)$. If $M_X(\Omega)$ is a **subincidence matrix** of $M_X(\Gamma)$, and every two rows r_i and r_k in $M_X(\Gamma)$ are joined by at least one scale *S* in $M_X(\Omega)$, then $M_X(\Omega)$ is called **spanning incidence matrix** of $M_X(\Gamma)$ and $M_X(\Omega)$ is called **spanning tree** of $M_X(\Gamma)$ if $M_X(\Omega)$ is a spanning and tree incidence matrix . The inverse of $M_X(\Gamma)$ is an incidence matrix of X^{-1} - Labeled graph Γ.

Now by direct calculations and the definitions above, we can prove the following results.

Lemma 2.8: If $M_X(\Gamma)$ is a tree incidence matrix of *X* – Labeled graph Γ with *n* rows, then $M_X(\Gamma)$ has *n* –1 columns.

3. Incidence matrices of directed graphs of finite groups.

Definition 3.1: An **incidence matrix of a directed graph of finite groups** consists of an incidence matrix of *X*-labeled graph $M_X(\Gamma)$ with a spanning tree matrix of *X*- labeled graph $M_X(T)$, and a base row $r^* = r_1$, together with a finite group G_r for each row *r*, and a finite group G_c for each column *c* , such that:

1) The columns of $M_X(\Gamma)$ are directed away from $r^* = r_1$;
2) Each column group G_c is a subgroup of $G_{i(c)}$;
3) Each column group G_c is embedded in $G_{t(c)}$ by a fixed monomorphism ψ_c, defined by $\psi_c(a) = y_c^{-1} a y_c$, $a \in G_c$, and $y_c =$ $s(c_j)$ is the non- zero entrance of c_j of $M_X(\Gamma)/M_X(T)$. It is denoted by $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$.

N.B.: Any incidence matrix of a graph of groups may be made into an incidence matrix of a directed graph of groups, that by choosing $M_X(T)$, a base row $r^* = r_1$, an orientation on columns and then identifying G_c with the image of $G_{i(c)}$ under the $M_X(\Gamma)$ relevant monomorphism.

For each directed column $c_j = (x_{ij}, x_{tj})$ in $M_X(\Gamma)$, let c^+ be c, and let $c^{-1} = (x_{tj}, x_{ij})$ be the inverse column with starting row $s(c_j^{-1}) = e(c_j)$, and $e(c_j^{-1}) = s(c_j)$, where, $x_{ij} = s(c_j) = y_j$, $x_{tj} = e(c_j) = y_j^{-1}$ and y_j are the entries of the column c_j, such that $y_j = 1$, if $c_j = (x_{ij}, x_{tj})$ is in $M_X(T)$.

N.B. We will denote y_{c_j} by y_j, (where, y_{c_j} are nonzero entries of $c_j = (y_{c_{ij}}, y_{c_{tj}})$, $y_{c_{ij}} \in X$, $y_{c_{tj}} \in X^{-1}$), such that y_{c_j} is equal to 1 or -1, if $c_j \in M_X(T)$.

Example: In this example, we will give a directed graph of groups and then, construct the incidence matrix of this directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$.

Let the directed graph of groups $(G_v, G_e, Y, T, v^*, \psi_e)$ be as follows:

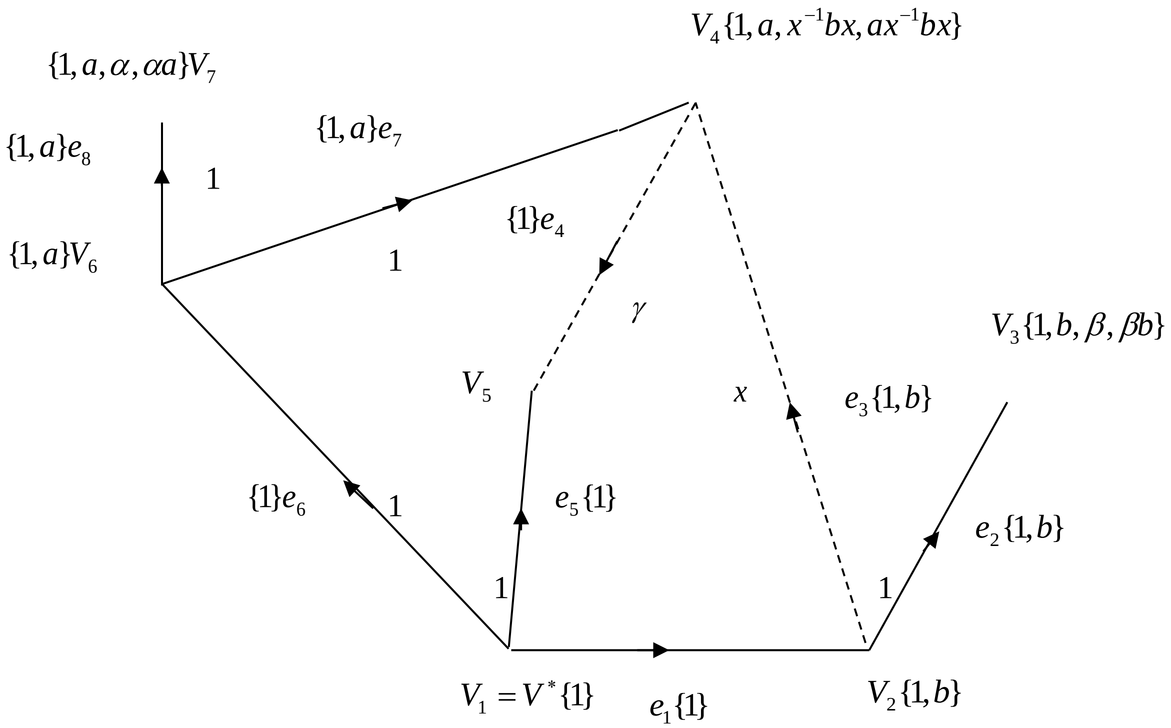

Figure 1. The directed graph of groups constructed in [3] p72 of the pregroup given in [4] p41.

The incidence matrix of the above directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$ is as below:

		$\{1\}$ e_1	$\{1,b\}$ e_2	$\{1,b\}$ e_3	$\{1\}$ e_4	$\{1\}$ e_5	$\{1\}$ e_6	$\{1,a\}$ e_7	$\{1,a\}$ e_8
$\{1\}$	r_1	1	0	0	0	1	1	0	0
$\{1,b\}$	r_2	-1	1	x	0	0	0	0	0
$\{1,b,,\beta,\beta b\}$	r_3	0	-1	0	0	0	0	0	0
$\{1,a,x^{-1}bx\ ,ax^{-1}bx\}$	r_4	0	0	x^{-1}	γ	0	0	-1	0
$\{1\}$	r_5	0	0	0	γ^{-1}	-1	0	0	0
$\{1,a\}$	r_6	0	0	0	0	0	-1	1	1
$\{1,a,\alpha,\alpha a\}$	r_7	0	0	0	0	0	0	0	-1

Figure 2. The incidence matrix of the directed graph of groups given in Figure 1 above.

4. The up-down pregroup of an incidence matrix of a directed graph of finite groups.

In this section we construct the up- down pregroup of the incidence matrix of a directed graph of groups as below;

Let $M_X(\Gamma)$ be the incidence matrix of a directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$.

The fundamental group of $\pi_1 = \pi_1(M_X(\Gamma))$ has the following presentation:
$< G_r, y_c; y_c^{-1} a y_c = \psi_c(a), \forall a \in G_c, y_c = 1 \forall c \in M_X(T), y_c \neq 1 \forall c \in M_X(Y)/M_X(T) >$. Now for each directed column $c \in M_X(Y)$, let *c* be also denoted by c^{+1} and let c^{-1} denote the inverse column with $s(c^{-1}) = e(c)$ and $e(c^{-1}) = s(c)$; also let y_{c_i} be of form $w = g_0.y_1^{\in_1}.g_1 y_2^{\in_2}.\cdots.y_n^{\in_n}.g_n$, where $\in_i = \pm 1$ and $\in_i = \pm 1$ and $c_1^{\in_1}, c_2^{\in_2}, \cdots, c_{n+1}^{\in_{n+1}}$ is a circuit at r^*, with rows $r^* = r_1, r_2, \cdots, r_{n+1} = r^*$ say, and where each g_i is in G_{r_i}. A word of this form and any subword of it is reduced if it contains no subword $y_c^{-1}.a.y_c$ or $y_c.\psi_c(a).y_c^{-1}$, where $a \in G_c$. If it does contain such a subword, we can , using the relations, substitute $\psi_c(a)$ or *a* respectively to obtain a shorter word of the given form representing the same element. Thus each element of the fundamental group is represented by a reduced word *w* of this form. Its inverse is representable by the word w^{-1} defined in the usual way. Moreover, by [13], the reduced word representing any element is unique modulo a succession of **interleaving**, i.e. substituting $\cdots g a^{-1}.y_c.\psi_c(a) h \cdots$ for $\cdots g.y_c.h.\cdots$ or vice – versa for any $a \in G_c$. Let $M_X(\Gamma)$ be the incidence matrix of the directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$ and let $q = c_1, c_2, \cdots, c_n$ be an upward scale in $M_X(\Gamma)$ which is a finite sequence of columns directed away of the base row $r*$. Let the rows of the scale *q* be $r^* = r_1, r_2, \cdots, r_n$. A word of type *q* is a word $w = g_1.y_1.g_2.y_2.\cdots, g_n.y_n.g_{n+1}$, where $g_i \in G_{r_i}$, $1 \leq i \leq n+1$, and every word *w* must be reduced and y_i is the non- zero entry of the starting row $s(c_i)$ of the column c_i. Now Let $q = c_1, c_2, \cdots, c_k$ and $q' = c_1', c_2', \cdots, c_h'$ be upward scales in $M_X(\Gamma)$ both starting at r^*. Let $w = g_1 y_1.g_2.y_2.g_3.\cdots.g_k.y_k.g_{k+1}$ and $w' = g_1'.y_1'.g_2'.y_2'.g_3'.\cdots.g_h'.y_{h+1}'$ be words of type *q* and q' respectively, where $k \leq h$. The word *w* is called an **initial subword** of the word w', written $w << w'$, if $c_j = c_j'$, and hence $y_{c_j} = y'_{c_j'}$, for $1 \leq j \leq k$, and if $g_j^{-1}.y_{j-1}^{-1}.g_{j-1}^{-1}.\cdots.y_1^{-1}.g_1^{-1}.g_1'.y_1'.\cdots.g_{j-1}'.y_j'.g_j'$ is an element of G_{r_j}, for each j.

Lemma 4.1: The relation " is an initial subword of " is both transitive and tree incidence matrix like, that is $w << w'$ and $w' << w''$, then $w << w''$, and if $w' << w$, $w'' << w$, then either $w' << w''$ or $w'' << w'$.

Proof: The result follows directly from the definition.

Now let $q = c_1, c_2, \cdots, c_k$ and $q' = c_1', c_2', \cdots, c_h'$ be upward scales in $M_X(\Gamma)$ both of them starting at r^* and ending at the same row *r*, and let $w = g_1 y_1.g_2.y_2.g_3.\cdots.g_k$ and $w' = g_1'.y_1'.g_2'.y_2'.g_3'.\cdots.g_h'$ be words of type *q* and q' respectively, such that the elements g_h' and g_k^{-1} are in G_r, and then such a word $w'w^{-1}$ is called an **up – down word**.

For example, the word $1.b.x.1.\gamma.1.1.1$ is an up-down word, from Figure 2 derived from the upward scales *q*, with rows r^* and r_5, and q', with rows r^*, r_2, r_4 & r_5.

Let $Q(M_X(\Gamma))$ be the set of all up-down words of the incidence matrix of a directed graph of groups $M_X(\Gamma)$. Reducing an up-down word in $M_X(\Gamma)$ gives another up-down. Therefore, we assume that such a word is reduced. We use $w'w^{-1}$ to denote an up-down word.

Lemma 4.2: Let $w'w^{-1}$ and $z'z^{-1}$ be reduced up-down words, then $(w'w^{-1})^{-1}(z'z^{-1})$ is in $Q(M_X(\Gamma))$ if and only if w' is an initial segment of z' or z' is an initial segment of w'.

Proof: Since the words $w'w^{-1}$ and $z'z^{-1}$ are both reduced, so reduction can only take place in the word $(w'w^{-1})^{-1}(z'z^{-1})$ between the last y_c of w'^{-1} and the first one of z'.

Moreover $(w'w^{-1})^{-1}(z'z^{-1})$ reduces to an up-down word if and only if either all y_c in w' or all y_c in z' are eliminated when putting $(w'w^{-1})^{-1}(z'z^{-1})$ in reduced form. This happens if and only if w' is an initial segment of z' or z' is an initial segment of w' respectively.

Now we show that $Q(M_X(\Gamma))$ is a pregroup. Since $Q(M_X(\Gamma))$ is a subset of $\pi_1=\pi_1(M_X(\Gamma))$, so $Q(M_X(\Gamma))$ satisfies conditions P_1, P_2 and P_4. It remains to be shown that $Q(M_X(\Gamma))$ satisfies condition P_5.

Define $L(w'w^{-1})=\{u'u^{-1};u'u^{-1}.w'w^{-1}\in Q(M_X(\Gamma))\}$, for $w'w^{-1}\in Q(M_X(\Gamma))$, as before.

Lemma 4.3: Let $w'w^{-1}$ and $z'z^{-1}$ be reduced up-down words, then $w' << z'$, implies that $w'w^{-1}\le z'z^{-1}$.

Proof: Suppose that $w' << z'$. To show $w'w^{-1}\le z'z^{-1}$, we must show that $L(z'z^{-1})$ is a subset of $L(w'w^{-1})$. If $(u'u^{-1})^{-1}(z'z^{-1})$ is in $Q(M_X(\Gamma))$, for some $u'u^{-1}\in Q(M_X(\Gamma))$, then , by Lemma 4.2, z' is an initial segment of u' or u' is an initial segment of z' (i.e. $z' << u'$ or $u' << z'$) respectively. Since $w' << z'$, in either case, by Lemma 4.1, we have $w' << u'$ or $u' << w'$) and then by Lemma 4.2 again, $(u'u^{-1})^{-1}(w'w^{-1})$ is defined in $Q(M_X(\Gamma))$. Therefore $L(z'z^{-1})$ is a subset of $L(w'w^{-1})$. Hence $w'w^{-1}\le z'z^{-1}$.

Theorem 4.4: $Q(M_X(\Gamma))$ is a pregroup.

Proof: To show $Q(M_X(\Gamma))$ is a pregroup, we will show that $Q(M_X(\Gamma))$ satisfies condition $P_5(iii)$ of Lemma 2.4. Therefore let $w'w^{-1}$ and $z'z^{-1}$ be reduced up-down words in $Q(M_X(\Gamma))$, and suppose that $(w'w^{-1})^{-1}(z'z^{-1})$ is defined in $Q(M_X(\Gamma))$. Hence by Lemma 4.2, we have $z' << w'$ or $w' << z'$. Thus by Lemma 4.3, $w'w^{-1}\le z'z^{-1}$ or $z'z^{-1}\le w'w^{-1}$. Therefore condition $P_5(iii)$ of Lemma 2.4 holds in $Q(M_X(\Gamma))$.

Definition 4.5: The set of all up – down words of the incidence matrix of a directed graph of groups $Q(M_X(\Gamma))$ is called **the up- down pregroup of** $(G_r,G_c,M_X(Y),M_X(T),r^*,\psi_c)$ **the Incidence matrix of a directed graph of groups**, where Q is the up- down pregroup of the directed graph of groups, as shown in [2] and [1].

Theorem 4.6: The Universal group of $Q(M_X(\Gamma))$ (it is denoted by $U(Q(M_X(\Gamma))$) is isomorphic to the fundamental group $\pi_1=\pi_1(M_X(\Gamma))$.

Proof: Since every element in $Q(M_X(\Gamma))$ is an element in $\pi_1=\pi_1(M_X(\Gamma))$, and since the tree incidence matrix $M_X(T)$ spans $M_X(Y)$ and is directed away from $r*$, every element of $\pi_1=\pi_1(M_X(\Gamma))$ can be written as a product of elements of $Q(M_X(\Gamma))$. Moreover the partial multiplication in $Q(M_X(\Gamma))$ implies the relations of $\pi_1=\pi_1(M_X(\Gamma))$.

5. An algorithm for the up-down Pregroup of incidence matrices of directed graphs of groups.

Let $M_X(\Gamma)=(G_r,G_c,M_X(Y),M_X(T),r^*,\psi_c)$ be the incidence matrix of a directed graph of groups, then we use the representation of the directed graph of groups of an up-down pregroup, to write down all the elements of the up-down pregroup of that graph of groups by applying the following algorithm. The steps are given below:

I) Find all up words $w=g_{i_1}.y_{j_1}.g_{i_2}.\cdots.g_{i_n}.y_{j_n}.g_{i_{n+1}}$ of type upward scales $q=(r^*=r_{i_1}),c_{i_1},r_{i_2}c_{i_2},r_{i_3},\cdots,r_{i_n},c_{i_n},r_{i_{n+1}}$, where $g_{i_k}\in G_{r_i}$, $1\le k\le n$ and y_{j_k} is the non- zero entrance of the row r_{i_k} which is the starting of the column c_{j_k}, as defined above and then proceed step II;

II) 1) If two up words $w=g_{i_1}y_{j_1}.g_{i_2}.\cdots.g_{i_n}.y_{j_n}.g_{i_{n+1}}$ and $w'=g'_{i_1}.y'_{i_1}.g'_{i_2}.\cdots.g'_{i_m}.y'_{j_m}.g'_{i_{m+1}}$, ending at the same row r_i, (i.e. row r_i contains non-zero entrances of forms x_{ij}^{-1} and x_{ik}^{-1}), then makes one of them an up word, say $w'=g'_{i_1}.y'_{i_1}.g'_{i_2}.\cdots.g'_{i_m}.y'_{j_m}.g'_{i_{m+1}}$ and makes the other up word $w=g_{i_1}y_{j_1}.g_{i_2}.\cdots.g_n.y_n.g_{n+1}$, down word by changing the direction of all g_i, columns and its entrance y_j to be $w^{-1}=g_{i_{n+1}}^{-1}.y_{j_n}^{-1}.g_{i_n}^{-1}.\cdots.y_{j_1}^{-1}.g_{i_1}^{-1}$, and by, identifying them we get an up-down word $w'.w^{-1}=g'_1.y'_1.g'_2.y'_2.g'_3.\cdots.g'_m.y'_m.g'_{i_{m+1}}.g_{i_n}^{-1}.y_{j_n}.\cdots.y_{j_1}^{-1}.g_{i_1}^{-1}$, (where, $g'_{i_{n+1}}=g_{i_{n+1}}.g_{i_{n+1}}^{-1}\in G_{r_{i_{n+1}}}$). Then proceed to step III;

2) If the up words $w = g_{i_1} y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}}$ end at an isolated row, then change the direction of all columns and its label to be $w^{-1} = g_{i_{n+1}}^{-1} .y_{j_n}^{-1} .g_{i_n}^{-1} .\cdots.y_{j_1}^{-1} .g_{i_1}^{-1}$, and by identifying them with the row r_i that both of them end with r_i, then we get an up-down word $w.w^{-1} = g_{i_1} y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}} . g_{i_{n+1}}^{-1} .y_{j_n}^{-1} .g_{i_n}^{-1} .\cdots.y_{j_1}^{-1} .g_{i_1}^{-1}$ $= g_{i_1} y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g'_{i_{n+1}} . .y_{j_n}^{-1} .g_{i_n}^{-1} .\cdots.y_{j_1}^{-1} .g_{i_1}^{-1}$, (where, $g'_{i_{n+1}} = g_{i_{n+1}} .g_{i_{n+1}}^{-1} \in G_{r_{i_{n+1}}}$). Then proceed to step III; III) If there is no other up-down word, then stop.

Proposition 5.1: All up words $w = g_{i_1} .y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}}$ of type upward scales in $M_X(\Gamma) = (G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$ are same as all up words $w = g_{i_1} .y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}}$ of type upward paths in $(G_r, G_c, Y, T, v^*, \psi_e)$.

Proof: Since all vertices *v* and edges *e* in $(G_r, G_c, Y, T, v^*, \psi_e)$ are represented by rows r and columns c in $M_X(\Gamma))$, and **associated** vertex groups G_v and edge groups G_e are represented by row groups G_r and columns groups G_c respectively, with entrances x_{ij} of the labeled y_e of the edges of $(G_r, G_c, Y, T, v^*, \psi_e)$ such that $y_e = 1$ if $e \in T$ and $y_e \neq 1$ if $e \in Y - T$. Therefore the direction and the labeling of columns of $M_X(\Gamma)$, are same as in $(G_r, G_c, Y, T, v^*, \psi_e)$. Hence all up words $w = g_{i_1} .y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}}$ of type upward scales in $M_X(\Gamma)$ are same as all up words $w = g_{i_1} .y_{j_1} .g_{i_2} .\cdots.g_{i_n} .y_{j_n} .g_{i_{n+1}}$ of type upward paths in $(G_r, G_c, Y, T, v^*, \psi_e)$.

Proposition 5.2: The algorithm must stop.

Proof: Since the size of $M_X(\Gamma)$ is $n \times m$ and all vertex groups and edge groups are finite, so $M_X(\Gamma)$ is finite incidence matrix. By step I, we get all reduced up- words, by step II we get all up- down reduced words, and then by step III, we will get all up- down reduced words. Since the origin X- labeled graph does not contain loops, so the set of all reduced up-down words is finite and then the algorithm must be stop after a finite time.

6. Conclusion

We have given a new application for the incidence matrices of X-labeled graphs. This application is the incidence matrices of directed graph of finite groups. Therefore, we have added certain conditions to allow the incidence of X-labeled graphs to be more confident with the definition of the directed graph of finite groups. By this way we can write a computer program to record all elements of the up- down pregroups of that the directed graphs of finite groups.

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