SQU Journal for Science, 2017, 22(1), 53-55 DOI: http://dx.doi.org/10.24200/squjs.vol22iss1pp53-55
2017 Sultan Qaboos University

Finite Dimensional Chebyshev Subspaces of ℓ_∞

Aref K. Kamal

Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, PC 123, Al-Khoud, Muscat, Sultanate of Oman, Email: akamal@squ.edu.om.

ABSTRACT: If A is a subset of the normed linear space X, then A is said to be *proximinal* in X if for each $x\in X$ there is a point $y_0\in A$ such that *the distance* between x and A; $d(x, A) = \inf\{\|x-y\|: y\in A\}= \|x-y_0\|$. The element y_0 is called a *best approximation* for x from A. If for each $x\in X$, the best approximation for x from A is unique then the subset A is called a *Chebyshev subset* of X. In this paper the author studies the existence of finite dimensional Chebyshev subspaces of ℓ_∞.

Keywords: Best approximation; Chebyshev subspaces; Banach lattice.

فضاءات تشيبيشيف الجزئية محدودة المدى في ℓ_∞

عارف كمال

الملخص: اذا كانت A مجموعة جزْئية من فضاء المتجهات المعياري X , و كان لكل x في X توجد نقطة y_0 في A بحيث ان المسافة d(x, A) بين x و A تساوي $\|x-y_0\|$ عندها نطلق على A اسم "مجموعة تقريبية" في X. النقطة y_0 تسمى "احسن تقريب" للنقطة x من A. اذا كان احسن تقريب لكل x في X من A وحيد عندها تسمى المجموعة A " مجموعة تشيبيشيفية جزئية" من X.

في هذه الورقة يدرس المؤلف امكانية وجود فضاءات جزئية تشيبيشيفية محدودة المدى في فضاء ℓ_∞

الكلمات المفتاحية: احسن تقريب ، فضاءات تشيبيشيف الجزئية و متشابكات باناخ.

1. Introduction

If A is a subset of the normed linear space X, then A is said to be *proximinal* in X if for each $x\in X$ there is a point $y_0\in A$ such that *the distance* between x and A; $d(x, A) = \inf\{\|x-y\|: y\in A\}= \|x-y_0\|$. In this case the element y_0 is called a *best approximation* for x from A. If for each $x\in X$, the best approximation for x from A is unique, then the subset A is called a *Chebyshev subset* of X. If Q is a compact Hausdorf topological space, then C(Q) denotes the Banach space of all continuous real valued functions defined on Q equipped with the *uniform norm*, that is, $\| f \| = \max\{| f(x) |: x\in Q\}$. For $1 \le p \le \infty$, ℓ_p denotes the classical Banach space of real sequences, and $L_p[0, 1]$ denotes the classical Banach spaces of real measurable functions.

Finite dimensional Chebyshev subspaces of Banach spaces have been the center of attention of mathematicians for a long time (see for example: [1-5]. One of their important properties is that the single valued metric projection function is continuous. (see, for example, [6]).

In 1956 Mairhuber [7] proved a special version of what was subsequently called Mairhuber's Theorem. Mairhuber's Theorem asserts that for any compact Hausdorff space Q, and for any $n \ge 2$, the Banach space C(Q) admits n dimensional Chebyshev subspaces if and only if Q is homeomorphic to a subset of a circle. ([8], Theorem 2.3, page 218). It was shown also that if Q is a compact Hausdorff space, then the n dimensional subspace N of C(Q) is a Chebyshev subspace if and only if each $g \ne 0$ in N has at most $n-1$ zeros. ([8] Theorem 2.2, page 215). In 1962, Ahiezer [9] showed that $L_1[0, 1]$ has no finite dimensional Chebyshev subspaces. It is easy to show that every finite dimensional subspace of a strictly convex space is a Chebyshev subspace ([10] page 23). Therefore for $1 < p < \infty$, every finite dimensional subspace of ℓ_p and every finite dimensional subspace of $L_p[0, 1]$ is a Chebyshev subspace.

In this paper the author studies the existence of the n dimensional Chebyshev subspaces of ℓ_∞. This is an important space of sequences, but it is not clear if it has any finite dimensional Chebyshev subspaces. In Section 2 it is shown that for n>1, this Banach space has no Chebyshev subspace of dimension n.

Before ending this section some terminologies and known results, that will be used later, will be mentioned.

Let ℓ_∞ denote the Banach space all real bounded sequences $x = (x_1, x_2, \ldots)$ equipped with the norm $\|x\|_\infty = \sup\{|x_i|: i = 1, 2, ..\}$. The Banach spaces X and Y are said to be isometric to each other if there is a linear mapping ψ from X onto Y such that $\|\psi(x)\| = \|x\|$ for each $x\in X$. It is clear that the isometry preserves the proximinality properties; that is, if ψ is an isometry from X onto Y and A is a subset of X, then for any $x\in X$, $d(x, A) = d(\psi(x), \psi(A))$. Therefore x_0 is a best approximation for x from A if and only if $\psi(x_0)$ is a best approximation for $\psi(x)$ from $\psi(A)$. **(** [11], page 143) shows that the space ℓ_∞ is not separable. In Theorem 1.1 there is another proof for this fact.

Theorem 1.1. : The space ℓ_∞ *is not separable.*

Proof: For each $0<\alpha<1$, let $0.\alpha_1\alpha_2\alpha_3\ldots$ be the binary representation of α, where $\alpha_i = 1$or 0 for all i = 1, 2, 3, … . Define $x_\alpha\in \ell_\infty$ by $x_\alpha = (\alpha_1, \alpha_2, \alpha_3\ldots)$. The set $A = \{x_\alpha;\ \alpha\in(0, 1)\}$ is an uncountable subset of ℓ_∞, and if $\alpha \neq \beta$ then $\|x_\alpha - x_\beta\|_\infty = 1$. Now let B be any dense subset of ℓ_∞, and let $\varepsilon = \frac{1}{3}$, then for any $\alpha\in (0, 1)$ one must have $B(x_\alpha, \varepsilon) \cap B = \{x\in B;\ \|x_\alpha - x\|< \frac{1}{3}\} \neq \phi$. For each $\alpha\in(0, 1)$ choose y_α to be any element in $B(x_\alpha, \varepsilon) \cap B$. It will be shown that if $\alpha \neq \beta$ in (0, 1) then $y_\alpha \neq y_\beta$. If this is true, then since the interval (0, 1) is uncountable and $\{y_\alpha;\ \alpha\in(0, 1)\} \subseteq B$, it follows that B is uncountable. Assume that $y_\alpha = y_\beta$ for some $\alpha \neq \beta$ in (0, 1), then $y_\alpha\in B(x_\alpha, \varepsilon) \cap B(x_\beta, \varepsilon)$. But then $\|x_\alpha - x_\beta\|_\infty \leq \|x_\alpha - y_\alpha\|_\infty + \|y_\alpha - x_\beta\|_\infty < \frac{1}{3} + \frac{1}{3} < 1$, which contradicts the fact that $\|x_\alpha - x_\beta\|_\infty = 1$.

Theorem 1.2. : *If Q is a compact subset of the circle, then C(Q) is separable.*

Proof: It is clear that the set of all polynomial with rational number coefficients is a countable dense subset of $C[0, 2\pi]$. So $C[0, 2\pi]$ is separable. Now let S be the unit circle in $R\times R$. It will be shown that C(S) is separable. If this is true, then for any compact subset Q of S, C(Q) must be separable. For each point $\Theta\in S$ there is a unique $\theta\in[0, 2\pi)$ such that $\Theta = (\cos\theta, \sin\theta)$. Define $\psi: C(S)\to C[0, 2\pi]$ by $\psi(f)(\theta) = f(\Theta)$ if $\theta \neq 2\pi$, and $\psi(f)(2\pi) = \psi(f)(0)$. It is clear that for each $f\in C(S)$, the function $\psi(f)$ is continuous on $[0, 2\pi]$. So ψ is well defined. It is also clear also that ψ is linear, and that $\|\psi(f)\| = \|f\|$ for each $f\in C(S)$. So ψ is an isometry from C(S) into $C[0, 2\pi]$. But $C[0, 2\pi]$ is separable, and therefore C(S) is also separable.

For a proof of a more general case one can refer to ([12] Proposition 7.6.2 page 126, and Proposition 623 page 95).

2. Main Results

Let X be a linear space and let $\leq$ be a partially ordered relation defined on X. Then $(X, \leq)$ is said to be a lattice if for each x and y in X, the least upper bound $x\vee y$ and the greatest lower bound $x\wedge y$ of x and y both exist in X. In this case if $x\in X$, then $|x|$ is defined to be; $|x| = x\vee -x$. The Banach space X is called a Banach Lattice if it is a lattice and for each x and y in X, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$. The element e in the Banach lattice is called a strong order unit if $\|e\| = 1$, and $x \leq e$ for all $x\in X$ with $\|x\| \leq 1$. The Banach lattice is called an Abstract M space if $\|x+y\| = \max\{\|x\|, \|y\|\}$ for each x and y in X satisfying that $x\wedge y = 0$. For more information about Banach Lattices one can refer to [13].

The following theorem is Theorem 4 page 59 of [14].

Theorem 2.1. : *Let X be a real Banach lattice. Then X is isometric to C(Q) for some compact Hausdorff space Q if and only if X is an abstract M space with a strong order unit.*

Theorem 2.2. : *The Banach space* ℓ_∞ *is an abstract M Banach Lattice with a strong order unit.*

Proof: Let $\leq$ be the relation defined on ℓ_∞ such that for each (x_i) and (y_i) in ℓ_∞, $(x_i) \leq (y_i)$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots$ Then $\leq$ is a partially ordered relation on ℓ_∞. If (x_i) and (y_i) are two elements in ℓ_∞ then the least upper bound, $(x_i)\vee(y_i)$, of (x_i) and (y_i) is $(x_i)\vee(y_i) = (\max\{x_i, y_i\})$, and the greatest lower bound is $(x_i)\wedge(y_i) = (\min\{x_i, y_i\})$. It is clear that if (x_i) and (y_i) are two elements in ℓ_∞ then both $(x_i)\vee(y_i)$ and $(x_i)\wedge(y_i)$ are also elements in ℓ_∞, and that if $|x_i| \leq |y_i|$ for all $i = 1, 2, \ldots$ then $\|(x_i)\|_\infty \leq \|(y_i)\|_\infty$. Therefore ℓ_∞ with the relation $\leq$ is a Banach lattice. If (x_i) and (y_i) are in ℓ_∞ and $(x_i)\wedge(y_i) = 0$ then $\min\{x_i, y_i\} = 0$ for all i =1, 2, … . Therefore $x_i \geq 0$ and $y_i \geq 0$. For each i =1, 2, .. , if $x_i > 0$ then $y_i = 0$, and if $y_i > 0$ then $x_i = 0$. Thus if $\min\{x_i, y_i\} = 0$, then $x_i + y_i = \max\{x_i, y_i\}$. So for any (x_i) and (y_i) in

ℓ_∞, if $(x_i)\wedge(y_i)$,= 0 then $\|(x_i) + (y_i)\|_\infty = \max\{\|(x_i)\|_\infty, \|(y_i)\|_\infty\}$. Thus ℓ_∞ is an abstract M space. Finally the constant function e = (e_1, e_2, …) defined by e_i =1 for each I = 1, 2, …, is a strong ordered unit for ℓ_∞.

Theorem 2.3. : *There is a compact Hausdorff space Q such that ℓ_∞ is isometric to C(Q).*

Proof: By Theorem 2.2, ℓ_∞ is an abstract M Banach Lattice with a strong order unit, and by Theorem 2.1, there is a compact Hausdorff Q such that X is isometric to C(Q).

The following theorem is an important theorem in Approximation Theory.

Theorem 2.4. (Mairhuber's Theorem): [15]: If n >1, and C(Q) admits an n-dimensional Chebyshev subspace, then Q is homeomorphic to a subset of the circle**.**

Theorem 2.5. : *If n >1, then ℓ_∞ has no Chebyshev subspace of dimension n.*

Proof: By Theorem 2.2, ℓ_∞ is isometric to C(Q) for some compact Hausdorff space Q. If this Q is homeomorphic to a subset of the circle, then by Theorem 1.2, ℓ_∞ is separable. But by Theorem 1.1, ℓ_∞ is not separable. Therefore, Q is not homeomorphic to a subset of the circle. By Theorem 2.4, if n >1, then ℓ_∞ has no n-dimensional Chebyshev subspace.

3. Conclusion

If X is the Banach space ℓ_∞ of all bounded sequences of real numbers then for n ≥ 2, X has no finite dimensional Chebyshev subspaces of dimension n.

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Received 17 May 2016
Accepted 1 December 2016