

Topo-Groups and a Tychonoff Type Theorem

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ABSTRACT: We define a topo-system on a group as a set of subgroups satisfying certain topology-like conditions. We study some fundamental properties of groups equipped with a topo-system (topo-groups) and using a suitable formalization of a subgroup filter, we prove a Tychonoff type theorem for the direct product of topo-compact topo-groups.

Keywords: Topo-groups; Topo-systems; Filters of subgroups; Subgroup ultra-filter

زمر- فضائية ونظرية من صنف تيخونوف

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الملخص: يتم تعريف نظام فضائي على الزمرة بأنه مجموعة من الزمر الجزئية التي تحقق شروط شبه فضائية محددة. سوف نقوم بدراسة بعض الخصائص الأساسية للزمر التي تملك نظام فضائي (زمر- فضائية) و باستعمال تشكيلات مناسبة من فلاتر الزمر الجزئية، سوف نبرهن نظرية من صنف تيخونوف والتي تتعلق بالاستتساخ الضربي للزمر الفضائية المتراسة فضائياً.

الكلمات المفتاحية: زمرة فضائية، نظم فضائية، فلاتر الزمر الجزئية، فلاتر فائقة للزمر الجزئية.



1. Introduction

In this article, we introduce an interesting topology-like concept concerning groups (which, by almost the same method, can be defined for other algebraic systems, see [1]). Given an arbitrary group G , we define a *topo-system* on G , as a set of subgroups satisfying certain conditions like a topology on a set. We will call such a group a *topo-group*. These topo-groups are not rare and as we will see, there are many examples of topo-groups. We investigate fundamental notions concerning topo-groups and we see that many basic concepts of topology can be introduced in the frame of topo-groups. A *filter of subgroups* of a group G will be defined in such a way that we will be able to formulate a Tychonoff type theorem for the direct product of *topo-compact* topo-groups. Throughout this article, $Sub(G)$ and $Sub^*(G)$ will be used for the set of subgroups and the set of non-identity subgroups of G , respectively. Also, for a set of subgroups $S = \{A_i\}_{i \in I}$, the notation $\langle A_i : i \in I \rangle$ represents the subgroup generated by S . For basic concepts of group theory, the reader can see [2].

Definition 1.1. Let G be a group and $T \subseteq Sub(G)$ be a set satisfying:

- 1 and $G \in T$,
- if $\{A_i\}_{i \in I}$ is a family of elements of T , then $\langle A_i : i \in I \rangle \in T$, and
- if A and B are elements of T , then $A \cap B \in T$.

Then we call T a topo-system on G and the pair (G, T) a topo-group. Elements of T will be called *T-open* subgroups.

Example 1.2. It is quite easy to find many examples of topo-groups. Here, we give a list of such examples:

- The set $T = Sub(G)$ is a topo-system and we call it the *discrete* topo-system on G .
- The set $T = \{1, G\}$ is a topo-system and we call it the *trivial* topo-system.

- 3- Let $B \subseteq G$ and $T_B = \{A \leq G : B \subseteq A\} \cup \{1\}$. This is the *principal* topo-system associated with B .
- 4- Let $T_{cf} = \{A \leq G : [G : A] < \infty\} \cup \{1\}$, where $[G : A]$ is the index of A inside G . This is the *cofinite* topo-system on G .
- 5- Let $T_n = \{A \leq G : A \text{ is normal in } G\}$. This is the *normal* topo-system on G .
- 6- Let $T_{char} = \{A \leq G : A \text{ is characteristic in } G\}$. This is the *characteristic* topo-system on G . Recall that A is a characteristic subgroup of G if it is invariant under every automorphism of G .
- 7- Let G be a topological group and T be the set of all open subgroups of G together with the identity subgroup. Then T is a topo-system on G .
- 8- Let V be a variety of groups and $T_V = \{A \in G : G/A \in V\} \cup \{1\}$. Then T_V is a topo-system on G and, in fact, we have $T_V = T_H \cap T_n$, where $H = O^V(G)$ is the V -residual of G (i.e., the smallest normal subgroup of G , such that G/H belongs to V).

Definition 1.3. A subgroup $A \leq G$ is *T-closed* if for all $x \in G \setminus A$, there exists $B \in T$ such that $x \in B$ and $A \cap B = 1$.

It is easy to verify that 1 and G are *T-closed*, and the intersection of any family of *T-closed* subgroups is *T-closed*. Let $X \leq G$ and $x \in X$. Suppose there is a *T-open* A with $x \in A \leq X$. Then we say that x is an *interior* element of X . The set of all interior elements of X is denoted by X° and it is called the interior of X . Clearly, the interior of X is a *T-open* subgroup. As an example, in the case of the normal topo-system (see the above example), we have $X^\circ = X_G$, the core of X (i.e., the maximal normal subgroup of G included in X). In the case of the co-finite topo-system, it can be verified that $X^\circ = 1$ or X . For any subgroup X , the *boundary* is $\partial X = X \setminus X^\circ$. A point $x \in G$ is a limit point of X , if for all *T-open* A containing x , we have $|A \cap X| \geq 2$. The *T-closure* of X is the subgroup generated by X and its limit points. Clearly, it is *T-closed*.

Unlike the ordinary topology, there is no symmetry between the concepts of *T-open* and *T-closed* subgroups. For example, if any subgroup of a topo-group G is *T-open*, we cannot say that any subgroup is *T-closed* as well. For example, consider the group $G = Z \times Z_4$ with the discrete topo-system. Then the subgroup $A = \langle(0,2)\rangle$ is not *T-closed*. In general, we have the following proposition.

Proposition 1.4. Let all cyclic subgroups of G be *T-closed*. Then all non-identity elements of G have prime orders.

Proof. Let $x \in G$ be a non-identity element. Then $A = \langle x^2 \rangle$ is *T-closed*. If x does not belong to A , then there is a $B \in T$ containing x such that $A \cap B = 1$. So, $\langle x \rangle \cap A = 1$ and hence $x^2 = 1$. If $x \in A$, then x has finite order n . Suppose $n = ab$. Then $\langle x^a \rangle$ is *T-closed* and hence

$$x \in \langle x^a \rangle \text{ or } \langle x^a \rangle \cap \langle x \rangle = 1. \tag{1}$$

In the first case we have $x^b = 1$ and in the second case, we have $x^a = 1$. So, n is a prime.

Definition 1.5. A subgroup $X \leq G$ is a *topo-compact* subgroup if any *T-open* covering of X has a finite sub-covering, i.e., if $X \subseteq \cup_{i \in I} A_i$, with $A_i \in T$, then there is a finite subset $J \subseteq I$, such that $X \subseteq \cup_{i \in J} A_i$. In the special case $X = G$, we say that G is *topo-compact*.

For example, a group is *topo-compact* with respect to the discrete topo-system if it is a finite union of cyclic subgroups. In fact, such a group is finite or infinite cyclic as the next proposition shows.

Proposition 1.6. Let (G, T) be *topo-compact* and suppose that T is the discrete topo-system. Then G is finite or infinite cyclic.

Proof. We have $G = \cup_{x \in G} \langle x \rangle$ and all $\langle x \rangle$ are *T-open*. So, by *topo-compactness*,

$$G = \langle x_1 \rangle \cup \langle x_2 \rangle \cup \dots \cup \langle x_n \rangle, \tag{2}$$

for some finite set of elements. Now by a theorem of Neumann (see [2], page 120), we may assume that any $\langle x_i \rangle$ has a finite index. Suppose G is infinite. Then all $\langle x_i \rangle$ are infinite as they have finite index, and hence G is torsion free. Assume $H \leq G$ to be a non-identity subgroup and $h \in H$ to be a non-identity element. Then, there is an index i such that $h \in \langle x_i \rangle$ and therefore $[\langle x_i \rangle : \langle h \rangle]$ is finite. So, already $[G : \langle h \rangle]$ is finite and hence H has finite index. A theorem of Fedorov (see [3], page 446) says that if any non-trivial subgroup of an infinite group has finite index, then the group is isomorphic to the infinite cyclic group Z . This completes the proof.

The concept of *topo-compact* groups may have many interesting interpretations if we consider different topo-systems. It can be verified that a *T-closed* subgroup of a *topo-compact* group is a *topo-compact* subgroup.

If (G, T) is a topo-group and $H \leq G$, then the set $S = \{A \cap H : A \in T\}$ generates a topo-system on H , which is an *induced* topo-system on H . We denote this topo-system by T_{ind} . We give a recursive construction of the elements of T_{ind} . Let $T_0 = S$ and for any ordinal α (see [4])

$$T_{\alpha+1} = \left\{ \bigcap_{i=1}^n \langle B_{ij} : j \in I_i \rangle : n \geq 1, \{B_{ij}\}_{j \in I_i} \subseteq T_\alpha \right\}$$

Also for limit ordinals we set

$$T_\alpha = \bigcup_{\beta < \alpha} T_\beta$$

Then it can be shown that $T_{ind} = \bigcup_a T_a$. Now, if (H, T_{ind}) is a topo-compact topo-group, then it can be proved that H is a topo-compact subgroup of G . The converse situation is very complicated because the T -open subgroups of (H, T_{ind}) have very complex structures.

Also if $H \in G$, then one checks that the set

$$T' = \left\{ \frac{AH}{H} : A \in T \right\}$$

is a topo-system on the quotient group G/H . So, we call it the quotient topo-system. If $\{(G_i, T_i)_{i \in I}\}$ is a family of topo-groups and $G = \prod_i G_i$, then the set

$$\left\{ \prod_i A_i : A_i \leq G_i, \text{ and almost all } A_i = G_i \right\}$$

is a topo-system on G . Note that, the situation here is much better than the case of ordinary product topology because of the following two trivial equalities:

1. $(\prod_i A_i) \cap (\prod_i B_i) = \prod_i (A_i \cap B_i)$
2. $\langle \prod_i A_{ij} : j \in J \rangle = \prod_i \langle A_{ij} : j \in J \rangle$

Definition 1.7. Let (G, T) and (H, S) be two topo-groups. A continuous map (or a *topomorphism*) is any homomorphism $f: G \rightarrow H$ with the property $f^{-1}(B) \in T$, for all $B \in S$. An invertible topomorphism with a continuous inverse is a *topeomorphism*.

Clearly, the projections $\pi_j: \prod_i G_i \rightarrow G_j$ are topomorphisms and also the natural map $q: G \rightarrow G/H$ is also a topomorphism. If H is a subgroup of a topo-group (G, T) , then the inclusion map $j: (H, T_{ind}) \rightarrow (G, T)$ is a topomorphism, too.

Remark 1.8. Let (G, T) be a topo-group. We can define a topology on G with the basis T . Let us denote this topology by T^* . A subset $X \subseteq G$ is open if X is a union of elements of T . This topology is never Hausdorff but G is compact if and only if G is topo-compact. For a subgroup $H \leq G$ with the induced topo-system T_{ind} , we have $(T^*)_{ind} \subseteq (T_{ind})^*$.

We close this section with two more examples of interesting topo-systems on arbitrary groups.

Example 1.9. Let G be a group and $H \leq K \leq G$. We define

$$T^{H,K} = \{A \leq G : [A, K] \subseteq H\} \cup \{G\}, \tag{3}$$

where $[A, K]$ is the commutator subgroup generated by all commutators $[a, b] = aba^{-1}b^{-1}$, with $a \in A$ and $b \in K$. Clearly, 1 and G belong to $T^{H,K}$. This set is closed under intersection, so let $\{A_i\}_{i \in I}$ be a family of elements of $T^{H,K}$. Suppose a_1, \dots, a_n are elements from the union of A_i 's and $x \in K$.

We have

$$[a_1 a_2 \dots a_n, x] = a_1 \dots a_n x a_n^{-1} \dots a_1^{-1} x^{-1}. \tag{4}$$

Now, for any $u \in K$, we have $a_i u a_i^{-1} u^{-1} \in H$, so for some $h_n \in H$,

$$[a_1 a_2 \dots a_n, x] = a_1 \dots a_{n-1} (h_n x) a_{n-1}^{-1} \dots a_1^{-1} x^{-1}.$$

Continuing this way, for some $h_{n-1} \in H$, we have

$$[a_1 a_2 \dots a_n, x] = a_1 \dots a_{n-2} (h_{n-1} h_n x) a_{n-2}^{-1} \dots a_1^{-1} x^{-1}. \tag{5}$$

Finally, we get

$$[a_1 a_2 \dots a_n, x] = h_1 h_2 \dots h_n \tag{6}$$

for some $h_1, \dots, h_n \in H$. This shows that $\prod_i A_i \in T^{H,K}$.

Example 1.10. Let G be group and $H \leq G$ be a fixed subgroup. Define

$$T_{conj}(H) = \{A \leq G : H \subseteq N(A)\},$$

where $N(A)$ is the normalizer of A in G . It can be verified that $T_{conj}(H)$ is a topo-system on G .

Hausdorff topo-groups

Suppose (G, T) is a topo-group and for any $x, y \in G$ with $\langle x \rangle \cap \langle y \rangle = 1$, there exist $A, B \in T$ with the properties

$$x \in A, y \in B, \text{ and } A \cap B = 1.$$

Then we say that (G, T) is a Hausdorff topo-system. A subgroup $A \leq G$ is *weak T-closed*, if for all x with $\langle x \rangle \cap A = 1$, there exists $B \in T$ such that $A \cap B = 1$. Recall that, in each ordinary Hausdorff topological space, every compact subset is closed. By a similar argument as in the ordinary topological spaces, one can see that the following proposition holds.

Proposition 2.1. *Let (G, T) be a Hausdorff topo-group and A be a topo-compact subgroup. Then A is weak T -closed.*

Clearly, any discrete topo-group is Hausdorff. If G is infinite and (G, T_{cf}) is Hausdorff, then for all non-identity $x, y \in G$, we have

$$\langle x \rangle \cap \langle y \rangle \neq 1 \tag{7}$$

so the intersection of any two non-trivial subgroups of G is non-trivial. Clearly, such a group is torsion free or a p -group for some prime. The groups Z , Q and Zp^∞ are examples of such groups. Torsion free non-abelian groups with this property are constructed by Adian and Olshanskii, [5]. An elementary argument shows that G can be embedded in Q if G is an R-group (i.e., a group satisfying $x^m = y^m \Rightarrow x = y$ for any non-zero m). The next proposition shows that the p -group case can be investigated by a very elementary argument, and so the hardest part of the problem is the case of torsion free groups which are not R-groups.

Proposition 2.2. *Let G be an infinite p -group which is Hausdorff with respect to the topo-system T_{cf} . Then G has a unique subgroup of order p . The converse is also true.*

Proof. Let G be a p -group and the intersection of any two non-trivial subgroups of G be non-trivial. Suppose x is a non-identity element of G . Then $\langle x \rangle$ has a subgroup A of order p . For any non-identity subgroup $B \leq G$, the intersection $A \cap B$ is non-trivial, so $A \subseteq B$, and hence A is a unique subgroup of G of order p . Conversely, suppose G has a unique subgroup of order p , say A . Then clearly, for any non-trivial subgroup B , we have $A \subseteq B$, and hence every two non-trivial subgroups of G have a non-trivial intersection.

Note that the subgroup A in the proof of the above proposition is *minimum* in the set of non-trivial normal subgroups of G . Therefore, by a well-known theorem of Birkhoff (see [1]), such a group G is *sub-directly irreducible*.

Corollary 2.3. *Any infinite p -group which is Hausdorff with respect to the topo-system T_{cf} is sub-directly irreducible.*

For any group G , the normal topo-group (G, T_n) is Hausdorff if and only if G satisfies

$$\langle x \rangle \cap \langle y \rangle = 1 \Rightarrow \langle x^G \rangle \cap \langle y^G \rangle = 1, \tag{8}$$

where $\langle x^G \rangle$ and $\langle y^G \rangle$ denote the normal closure of $\langle x \rangle$ and $\langle y \rangle$, respectively. In the next theorem, we give some properties of this kind of groups. Note that a Q-group is a group in which every element x has a unique m -th root $x^{1/m}$, for every non-zero integer m .

Theorem 2.4. *Let G be a torsion free group which is Hausdorff with respect to the topo-system T_n . Then for any $x, u \in G$, there are non-zero integers m and k such that $ux^k u^{-1} = x^m$. Further, if G is also a Q-group, then there exists a family of normal subgroups A_i , such that*

- 1- $G = \bigcup_i A_i$,
- 2- $A_i \cong Q$,
- 3- $A_i \cap A_j = 1$.

As a result, G is abelian.

Proof. Let $A = G \setminus 1$ and define a binary relation on A by

$$x \equiv y \Leftrightarrow \langle x \rangle \cap \langle y \rangle \neq 1.$$

This is an equivalence relation on A . Let $E(x)$ be the equivalence class of x and $\{E_i = E(x_i) : i \in I\}$ the set of all such classes. Then we have

$$G = \bigcup_i \bigcup_{y \in E_i} \langle y \rangle$$

Now, since

$$\langle x \rangle \cap \langle y \rangle = 1 \Rightarrow \langle x^G \rangle \cap \langle y \rangle = 1, \tag{9}$$

so,

$$\langle x^G \rangle \subseteq \bigcup_{y \in E(x)} \langle y \rangle.$$

Let $u \in G$ be arbitrary. Then there is $y \in E(x)$ such that $uxu^{-1} \in \langle y \rangle$. Hence, for some i , m and k we have

$$uxu^{-1} = y^i, \quad x^m = y^k.$$

Therefore, for any $x, u \in G$, there are non-zero integers m and k such that $ux^k u^{-1} = x^m$. Now, suppose G is a Q-group. Then, we have

$$\begin{aligned} E_i &= \{y \in G : \exists m, k \text{ (non-zero)} \ x^m = y^k\} \\ &\subseteq \{x_i^\alpha : \alpha \in Q\} \\ &\subseteq G. \end{aligned}$$

Suppose $A_i = \{x_i^\alpha : \alpha \in Q\}$. Then, any A_i is a subgroup of G and clearly, we have 1-2-3. Note that, since for any $u \in G$ we have $ux_i u^{-1} = x_i^\alpha$, for some $\alpha \in Q$, so A_i is a normal subgroup. Now, $[A_i, A_j] \subseteq A_i \cap A_j$, so, by 3, we have $[A_i, A_j] = 1$. This shows that G is abelian.

We can use the Malcev completion of torsion free locally nilpotent groups (see [6] and [7]) to prove the next result on Hausdorff groups with respect to T_n .

Theorem 2.5. *Let G be a torsion free locally nilpotent group which is Hausdorff with respect to T_n . Then G is abelian.*

Proof. The same argument as above shows that for any $x, u \in G$, there are non-zero integers m and k such that $ux^k u^{-1} = x^m$. Now, suppose G^* is the Malcev completion of G . We know that G^* is a Q-group. By the notations of the above proof, let $A_i = \{x_i^\alpha : \alpha \in Q\}$, which is a subgroup of G^* . A similar argument shows that

- 1 - $G = \bigcup_i A_i$,
- 2 - $A_i \cong Q$,
- 3 - $A_i \cap A_j = 1$.

Since $x_i x_j x_i^{-1} = x_j^\alpha$, for some $\alpha \in Q$, so $[x_i, x_j] \in A_i$ and similarly it also belongs to A_j . This shows that x_i and x_j are commuting. Therefore, we have $[x_i^p, x_j^q] = 1$ for all $p, q \in Q$. So, $[A_i, A_j] = 1$, which shows that G is abelian.

Filters of Subgroups

For standard notions of filter theory, the reader could use [8]. A *filter of subgroups* (or a subgroup filter) is any subset $F \subseteq \text{Sub}^*(G)$, with the following properties:

- a) $G \in F$.
- b) If $A \leq B \leq G$ and $A \in F$, then $B \in F$.
- c) If A and $B \in F$, then $A \cap B \in F$.

Example 3.1. For any group G , the sets $\{G\}$ and $\text{Sub}^*(G)$ are trivial examples of subgroup filters. For each $x \in G$, there exists the *principal* filter of subgroups

$$F_x = \{A \in \text{Sub}^*(G) : x \in A\}.$$

If G is an infinite group, then the set

$$F_{cf} = \{A \leq G : [G : A] < \infty\}$$

is the *co-finite* filter of subgroups.

Like ordinary filters, any filter of subgroups has *finite intersection property* ("fip" for short):

$$A_1, \dots, A_n \in F \Rightarrow \bigcap_{i=1}^n A_i \neq 1.$$

On the other hand, a subset $S \subseteq \text{Sub}^*(G)$ with fip is contained in a unique minimal filter of subgroups (which is the subgroup filter generated by S); if we define

$$S_* = \left\{ \bigcap_{i=1}^n A_i : n \geq 1 \text{ and } A_i \in S \right\},$$

then the subgroup filter generated by S is equal to

$$F_S = \{B \leq G : A \subseteq B \text{ for some } A \in S_*\}.$$

Let $P(G)$ be the power set of G and $F_1 \subseteq P(G)$ be an ordinary filter. Then clearly the set $F = F_1 \cap \text{Sub}^*(G)$ is a filter of subgroups. The converse is also true:

Proposition 3.2. *Let $F \subseteq \text{Sub}^*(G)$ be a subgroup filter. Then there is an ordinary filter $F_1 \subseteq P(G)$ such that $F = F_1 \cap \text{Sub}^*(G)$.*

Proof. Let

$$F_1 = \{X \subseteq G : A \subseteq X \text{ for some } A \in F\}.$$

It can be easily verified that F_1 is an ordinary filter and $F = F_1 \cap \text{Sub}^*(G)$.

Definition 3.3. An *ultra-filter* of subgroups is a subgroup filter $F \subseteq \text{Sub}^*(G)$ such that for any finite set of subgroups A_1, \dots, A_n , the condition $\bigcup_i A_i \in F$ implies $A_i \in F$, for some i .

Proposition 3.4. *Let $F_1 \subseteq P(G)$ be an ordinary ultra-filter. Then $F = F_1 \cap \text{Sub}^*(G)$ is an ultra-filter of subgroups.*

Proof. Suppose $A_1 \cup A_2 \cup \dots \cup A_n \in F$, where A_1, A_2, \dots, A_n are subgroups. Suppose by contrary that none of A_i s belong to F . Then, clearly, none of A_i s belong to F_1 , and as F_1 is an ordinary ultra-filter, we have $G \setminus A_1, G \setminus A_2, \dots, G \setminus A_n \in F_1$. This means that

$$\bigcap_{i=1}^n (G \setminus A_i) \in F_1$$

and hence,

$$\emptyset = (\bigcap_{i=1}^n (G \setminus A_i)) \cap (\bigcup_{i=1}^n A_i) \in F_1 \tag{10}$$

which is a contradiction.

Proposition 3.5. *Let F be a subgroup filter on G . Then there exists an ultra-filter of subgroups containing F .*

Proof. We have $F = F_1 \cap \text{Sub}^*(G)$ for some ordinary filter F_1 , but there exists an ordinary ultra-filter F_2 such that $F_1 \subseteq F_2$. Now, $F^* = F_2 \cap \text{Sub}^*(G)$ is a subgroup ultra-filter containing F .

Definition 3.6. Let G and H be two groups and $F \subseteq \text{Sub}^*(G)$ be a filter of subgroups. Let $f : G \rightarrow H$ be a group homomorphism and define

$$f_*(F) = \{A \leq H : f^{-1}(A) \in F\}. \tag{11}$$

One can see that this is a subgroup filter on H and further, it is a subgroup ultra-filter, if F is so.

The following definition connects two notions of topo-groups and ultrafilters of subgroups.

Definition 3.7. Let (G, T) be a topo-group and $F \subseteq \text{Sub}^*(G)$ be a subgroup ultra-filter. Let $y \in G$. We say that F converges to y , if for all $A \in T$ with $y \in A$, we have $A \in F$. In this situation we write $F \rightarrow y$.

Note that if (G, T) and (H, S) are topo-groups, and if $f : G \rightarrow H$ is a topomorphism, then for an ultra-filter of subgroup F , the condition $F \rightarrow x$ implies $f_*(F) \rightarrow f(x)$, for all $x \in G$. We are now ready to prove the main theorem of this section:

Theorem 3.8. *Let (G, T) be a topo-group. Then G is topo-compact if and only if any ultra-filter of subgroups converges to some point in G .*

Proof. Let G be topo-compact and F be an ultra-filter of subgroups on G . Suppose by contrary that F does not converge to any element of G . So, for all $y \in G$, there exists a T -open subgroup $A_y \leq G$ such that $y \in A_y$ and A_y does not belong to F . We have $G = \bigcup_y A_y$, so G can be covered by a finite number of A_y 's, say A_{y_1}, \dots, A_{y_n} . Hence

$$A_{y_1} \cup \dots \cup A_{y_n} = G \in F$$

and therefore $A_{y_i} \in F$, for some i , since F is an ultra-filter. This is a contradiction, so F has at least one convergence point in G .

Now, suppose that G is not topo-compact. Hence, there is a covering $G = \bigcup_{i \in J} A_i$, where each A_i is T -open and $G \neq \bigcup_{i \in J'} A_i$ for any finite subset $J' \subseteq J$. This means that the set

$$S = \{G \setminus A_i : i \in J\} \tag{12}$$

has 'set-fip' (i.e., the intersection of any finite number of its elements is non-empty) and so there exists an ordinary ultra-filter F_1 containing S . Let $F = F_1 \cap \text{Sub}^*(G)$. Then F is a subgroup ultra-filter. For any $y \in G$, we have $y \in A_i$ for some i . But as F_1 is an ultra-filter, A_i does not belong to F_1 and already it does not belong to F . This shows that F does not converge to y .

We say that two elements x and y in a group G are *cyclically distinct* if $\langle x \rangle \cap \langle y \rangle = 1$. So, the above theorem says that a topo-system is topo-compact if and only if every ultra-filter of subgroups has at least one point of convergence up to the cyclic distinction. A similar assertion holds for Hausdorff topo-groups.

Theorem 3.9. *A topo-group (G, T) is Hausdorff if and only if any ultra-filter of subgroups in G converges to at most one point (up to cyclic distinction).*

Proof. Let F be an ultra-filter of subgroups in G and (G, T) be Hausdorff. Let x and y be cyclically distinct but $F \rightarrow x$ and in the same time $F \rightarrow y$. We know that there are $A, B \in T$ containing x and y , respectively and $A \cap B = 1$. But then we have also $A, B \in F$ and so $1 = A \cap B \in F$, which is impossible.

Conversely, let (G, T) be not Hausdorff. Then there are cyclically distinct x and y , such that for all T -open subgroups A and B with $x \in A$ and $y \in B$, we have $A \cap B \neq 1$. Suppose $S = F_x \cup F_y$. Then clearly S has fip and hence there exists an ultra-filter of subgroups, say F , such that $S \subseteq F$. Now suppose $A \in T$ and $x \in A$. Then $A \in S \subseteq F$ and therefore $F \rightarrow x$. Similarly, we have $F \rightarrow y$.

A Tychonoff Type Theorem

It seems that many known theorems of topology have versions in topo-groups. Once we prove such a theorem, it is possible to translate it into the language of groups and find an interesting theorem of group theory. In this section, we prove an analogue of the compactness theorem of Tychonoff (see [8]) for topo-groups.

Theorem 4.1. *Let $\{(G_i, T_i)\}_{i \in I}$ be a family of topo-compact topo-groups. Then $G = \prod_i G_i$ is also topo-compact.*

Proof. Let $F \subseteq \text{Sub}^*(G)$ be a subgroup ultra-filter. We prove that it converges to at least one point in G . Note that $(\pi_i)_*(F)$ is an ultra-filter of subgroups in G_i and since G_i is topo-compact, so it converges to some point $x_i \in G_i$. Suppose $x = (x_i) \in G$. We prove that $F \rightarrow x$. Let $A \leq G$ be T -open and $x \in A$. We may assume that $A = \prod_i A_i$, with $A_i \in T_i$ and such that almost all $A_i = G_i$. Since $x_i \in A_i$, so $A_i \in (\pi_i)_*(F)$, i.e. $\pi_i^{-1}(A_i) \in F$. Now, clearly

$$A = \bigcap_i \pi_i^{-1}(A_i),$$

and the intersection consists of finitely many non-trivial elements of F , so it belongs to F and this shows that $F \rightarrow x$, therefore G is topo-compact.

Conflict of interest

The author declares no conflict of interest.

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References

1. Burris, S. and Sankappanavar, H.P. *A course in universal algebra*, The Millennium Edition, 2012 update.
2. Scott, W.R. *Group theory*, Dover, 1987, Springer-Verlag, New York, Heidelberg, Berlin, New Jersey.
3. Passman, D.S. *The algebraic structure of group rings*, Wiley, New York, London, Sydney, 1977.
4. Thomas, J. *Set theory*, Springer, 2006, 73-90, Berlin, Heidelberg.
5. Olshanskii, A.Y. *Geometry of defining relations in groups*, Kluwer Academic Publishers, 1991, 296-307.
6. Majewicz, S. On classes of exponential A-groups, *Comm. Alg.*, 2010, **38(4)**, 1363-1384.
7. Temple, H.F. Remarks on the Malcev completion of torsion free locally nilpotent groups, *Cahiers de Topologie et Géométrie*, 1994, **350(1)**, 75-84.
8. Wilansky, A. *Topology for analysis*, Dover publication, New Jersey, 2008, 121-134.

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