SQU Journal for Science, 2023, 28(2),44-55 DOI:10.53539/squjs.vol28iss2pp44-55
Sultan Qaboos University

Analysis of Fractional Linear Multi-Step Methods of Order Four from Super-convergence

Khadija Al-Hassani and H.M. Nasir*

Department of Mathematics, Sultan Qaboos University, P.O.Box: 36, Al Khod, Muscat, Sultanate of Oman. *Email: nasirh@squ.edu.om

ABSTRACT: We analyzed two implicit fractional linear multi-step methods of order four for solving fractional initial value problems. The methods are derived from the Gr̈unwald-Letnikov approximation of fractional derivative at a non-integer shift point with super-convergence. The weight coefficients of the methods are computed from fundamental Gr̈unwald weights, making them computationally efficient when compared with other known methods of order four. We also show that the stability regions are larger than those of the fractional Adams-Moulton and fractional backward difference formula methods. We present numerical results and illustrations to verify that the theoretical results obtained are indeed satisfied.

Keywords: Fractional initial value problem; Generating function; Super-convergence; Grunwald-Letnikov approximation; Fractional linear multi-step method; stability region.

تحليل طرق كسرية خطية متعددة الخطوات ذات الرتبة الرابعة من التقارب الفائق

خديجة الحسني وحنيفة محمد ناصر

الملخص: في هذا البحث نقوم بتحليل طريقتين ضمنيتين كسريتين خطيتين ومتعددتي الخطوات ذات الرتبه الرابعه لحل مسائل القيم الأولية الكسرية، تم إشتقاق الطرق من تقريب جرنوالد للمشتقة الكسرية عند نقطة تحول عدد غير صحيح بإستخدام التقارب الفائق، حيث يتم حساب معاملات الوزن للطرق من أوزان جرنوالد لنكتوف الأساسية مما يجعلها فعالة من الناحية الحسابية عند مقارنتها بالطرق الأخرى المعروفة من الترتيب الرابع، كما وضحنا أيضا أن مناطق الإستقرارلهذه الطرق أكبر من مناطق أدمز مولتن الكسرية وكذلك طرق صيغة الفرق العكسي الجزئي، تم أيضا تقديم النتائج العددية والرسوم التوضيحية للتحقق من أن النتائج النظرية التي تم الحصول عليها مرضية بالفعل.

الكلمات المفتاحية: مسائل القيم الأولية الكسرية؛ الدالة المولدة؛ التقارب الفائق؛ تقريب جرنوالد لنكتوف؛ طرق أدمز مولتن الكسرية؛ مناطق الإستقرار.

1. Introduction

Consider the fractional initial value problem (FIVP)

$$D_{t_0}^{\beta} y(t) = f\big(t, y(t)\big), \quad t \geq t_0, \quad 0 < \beta \leq 1, \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where $D_{t_0}^{\beta}$ is the left Caputo fractional derivative operator defined in Section 2, $f(t, y)$ is a function satisfying the Lipschitz condition in the second argument y which guarantees a unique solution to the problem [1,2]. When $\beta = 1$, problem (1) with (2) becomes the classical initial value problem(IVP) with first order derivative.

Many numerical schemes for approximately solving the FIVP (1) have been proposed in the recent past. The numerical methods referred to as fractional linear multi-step methods (FLMMs) are of particular interest.

The simplest and most highly investigated FLMM is the fractional Euler method (also known as the Grünwald-Letnikov method) obtained by approximating the fractional derivative $D_{t_0}^{\beta} y(t)$ in (1), after some modifications, by the Grunwald-Letnikov (GL) approximation (See Section 2).

Converting the FIVP (1) in the form of Volterra integral equation (VIE) of the second kind, Lubich [3—5] proposed a class of higher-order FLMMs for the VIE as convolution quadrature rules. The quadrature coefficients of the FLMMs are obtained from the fractional order power of some rational polynomials from the linear multistep methods(LMMs) for classical IVPs. Moreover, Lubich [5], suggested a set of implicit fractional backward difference formula (FBDF) methods as a subclass of these FLMMs.

Galeone and Garrappa [6] investigated another subclass of implicit FLMMs and called them fractional Adams-Moulton methods (FAM), also suggested by Lubich in [5]. They also constructed in [7,8] some explicit FLMMs of this subclass and called fractional Adams-Bashforth methods (FAB). Another set of explicit FLMMs was constructed by Bonab and Javidi [9].

Aceto [10] constructed another subclass of FLMMs by approximating Lubich's generating functions of the FBDFs by Pade approximations. However, in this class, the orders of the FLMMs are reduced compared to the source FLMMs.

The present authors proposed two new implicit FLMMs of order 4 with preliminary properties and tests presented in [11]. The methods use the super convergence of the GL approximation. Earlier, the authors used super convergence to derive an FLMM of order 2 in [12].

This paper analyzes the two implicit FLMMs of order 4 presented in [11]. As FLMMs of orders higher than two are not A-stable according to Dahlquist's second barrier for FIVPs [13], we analyze the stability of the methods through $A(\pi/2)$-stability and unconditional stability. We also show that the methods are better in stability than the FAM4 of order 4 and one of the methods is better than FBDF4 of order 4.

The computational costs of these methods have also been compared with other order 4 methods and show that the new FLMMs are computationally competitive with the FAM and simpler than that the FBDF4.

This paper is organized as follows. Section 2 gathers the necessary definitions, theories, and results on factional calculus and numerical solutions of FIVPs. In Section 3, we give the main results by constructing the new FLMMs along with an algorithm to compute approximate solutions using these methods. In Section 4, we analyze the stability of the methods. Section 5 compares our methods with other known methods of order 4. In Section 6, we draw conclusions.

2. Preliminaries

The fundamental definitions of fractional derivatives in fractional calculus are typically presented as follows:

Definition 2.1 Let $y(t)$ be a function defined in the interval domain $[t_0, T]$ and is sufficiently smooth to hold the following:

1. When $y \in L_1([t_0, T])$, the Riemann-Liouville (RL) fractional integral of order $\beta > 0$ of $y(t)$ is defined as

$$J_{t_0}^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} y(s) ds. \tag{3}$$

2. The RL fractional derivative of order $\beta > 0$ is defined by

$$\widehat{D}_{t_0}^{\beta} y(t) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dt^m} \int_{t_0}^{t} (t-s)^{m-\beta-1} y(s) ds, \qquad m-1 \le \beta < m. \tag{4}$$

where $m = \lceil \beta \rceil$ is the integer ceiling of β, and $\Gamma(\cdot)$ denotes Euler's gamma function. $D^m = \frac{d^m}{dt^m}$ is the m-th order differential operator.

3. The Caputo derivative of order $\beta > 0$ is defined by

$$D_{t_0}^{\beta} y(t) = \frac{1}{\Gamma(m-\beta)} \int_{t_0}^{t} (t-s)^{m-\beta-1} y^{(m)}(s) ds, \qquad m-1 \le \beta < m. \tag{5}$$

4. The Grünwald-Letnikov definition of fractional derivative is given by

$$\widetilde{D}_{t_0}^{\beta} y(t) = \lim_{\tau \to 0} \frac{1}{\tau^{\beta}} \sum_{k=0}^{\infty} g_k^{(\beta)} y(t-k\tau), \tag{6}$$

where $g_k^{(\beta)} = (-1)^k \frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)k!}$ are called the *Grünwald weights*.

5. A shifted form of GL fractional derivative is also available [14]:

$$\widetilde{D}_{t_0,r}^{\beta} y(t) = \lim_{\tau \to 0} \frac{1}{\tau^{\beta}} \sum_{k=0}^{\infty} g_k^{(\beta)} y(t-(k-r)\tau), \tag{7}$$

where r is the shift which is often taken to be an integer, but any real shift is valid.

Remark: The fractional derivatives given above are called the *left* fractional derivatives as there are also their *right* counterparts. For details see [15,16] and the references therein.
The fractional integrals and derivatives are related such that the RL and Caputo derivatives are two left inverses of the RL integral [1]:

$$\hat{D}_{t_0}^{\beta} J_{t_0}^{\beta} y(t) = D_{t_0}^{\beta} J_{t_0}^{\beta} y(t) = y(t). \tag{8}$$

However, the two fractional derivatives in (4) and (5) are related by

$$D_{t_0}^{\beta} y(t) = \hat{D}_{t_0}^{\beta} [y(t) - T_{m-1}(t - t_0).,\quad y \in C^{m-1}[t_0, T],\qquad y^{(m)} \in L^1[t_0, T],$$

where

$$T_{m-1}(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(t_0).$$

Hence, the RL and Caputo fractional derivatives are equal under the homogeneous initial conditions $y^{(k)}(t_0) = 0, k = 0,1,\dots, m-1$ [1, 2]. The GL fractional derivative in (6) and its shifted counterpart in (7) are also equivalent to the Caputo derivative under homogeneous conditions and are often utilized as tools for numerical approximations of fractional derivatives.

2.1 Approximation of fractional integrals and derivatives

To approximate the fractional integrals and derivatives, the involving domain $[t_0, T]$ is discretized into a computational domain with uniformly spaced discrete points $t_k = t_0 + k\tau, k = 0,1,\dots, N$ with a fixed step size $\tau = (T - t_0)/N$. The fractional integral can be approximated by using a quadrature rule as the sum of weighted function values at the discrete points of the involved integrating domain. Common quadrature rules used in this sense are the rectangular and trapezoid rules [17-19]. Lubich [5] introduced a convolution quadrature approximation formula for the fractional integral

$$J_{\tau}^{\beta} y(t) = \tau^{\beta} \sum_{k=0}^{n} \omega_k y(t - k\tau),$$

where the weights ω_k are obtained from the power series expansion of the generating function $\omega(\xi) = \left(\frac{\sigma(1/\xi)}{\rho(1/\xi)}\right)^{\beta}$ with (ρ, σ) being the pair of generating polynomials of the LMM for classical IVPs [1]. The order of consistency for the FLMM is the same as that of the underlying LMM. As for the approximation of fractional derivatives, the fundamental approximation for the RL fractional derivative is obtained from the GL (or generally the shifted GL) definition by simply dropping the limit for a fixed step size τ. This gives an order one approximation $\delta_{\tau,r}^{\beta} y(t)$ with an integer shift r [5,21].

$$\delta_{\tau,r}^{\beta} y(t) = \frac{1}{\tau^{\beta}} \sum_{k=0}^{\infty} g_k^{(\beta)} y(t - (k - r)\tau) = D_{t_0}^{\beta} y(t) + O(\tau),$$

where the initial value y_0 has been subtracted from $y(t)$ to satisfy the homogeneous initial condition so that the different definitions coincide.
However, for the particular non-integer shift $r = \beta/2$, the above Grünwald approximation gives order 2 displaying super convergence [22].

$$\delta_{\tau,\beta/2}^{\beta} y(t) = D_{t_0}^{\beta} y(t) + O(\tau^2).$$

Analogous to the convolution quadrature approximation for fractional integral, fractional derivatives can also be approximated by convolution quadrature formula in a similar form

$$\Omega_{\tau,r}^{\beta} y(t) = \tau^{-\beta} \sum_{k=0}^{\infty} w_k y(t - k\tau),$$

where w_k are the coefficients of a generating function $W(\xi)$.
The order of consistency of an FLMM can be obtained from its generating function through the following theorem.

Theorem 2.2 [14,23,24] *Let $W(\xi)$ be the generating function of an approximation in the shifted form of the fractional derivative $D_{t_0}^{\beta} y(t)$,*

$$\Omega_{\tau,r}^{\beta} y(t) = \frac{1}{\tau^{\beta}} \sum_{k=0}^{\infty} w_k y(t - (k - r)\tau,$$

where $y(t)$ is sufficiently smooth. The order of the shifted approximation with shift r is m if and only if

$$G(x) = \frac{1}{x^{\beta}} W(e^{-x}) e^{rs} = 1 + O(x^m). \tag{9}$$

Moreover, we have

$$\Omega_{\tau,r}^{\beta} y(t) = D_{t_0}^{\beta} y(t) + \tau^m a_p D_{t_0}^{\beta+m} y(t) + \tau^{m+1} a_{p+1} D_{t_0}^{\beta+m+1} y(t) + \cdots, \tag{10}$$

where $\boldsymbol{a_k} \equiv \boldsymbol{a_k(\beta, r)}$ are the coefficients of the series expansion of $\boldsymbol{G(x)}$:

$$\boldsymbol{G(x) = 1 + \sum_{k=p}^{\infty} a_k\, x^k}.$$

2.2 FLMM scheme

The general form of an FLMM scheme for the FIVP in (1) and (2) is given by

$$\sum_{k=0}^{n} p_k y_{n-k} = \tau^{\beta} \sum_{k=0}^{n} q_k f_{n-k}, \tag{11}$$

where p_k and q_k are the coefficients of the generating functions

$$p(\xi) = \sum_{k=0}^{\infty} p_k \xi^k \quad \text{and} \quad q(\xi) = \sum_{k=0}^{n} \sigma_k \xi^k.$$

and y_k and f_k denote

$$y_k \approx y(t_k) \quad \text{and} \quad f_k = f(t_k, y_k). \tag{12}$$

In numerical computations using the FLMMs of order more than one, the intended order m is achieved only for a certain class of functions, specifically for functions of the form $y(t) = t^{\alpha} g(t)$, $\alpha \geq m$, with $g(t)$ analytic [5]. However, for $\alpha < m$, the order is reduced to $O(h^{\alpha})$ only. To remedy this order reduction, an additional sum is introduced in (11) to have the approximation scheme

$$\sum_{k=0}^{s} w_{n,k} y_k + \sum_{k=0}^{n} w_k^{(\beta)} y_{n-k} = \tau^{\beta} \sum_{k=0}^{n} \sigma_k f_{n-k}. \tag{13}$$

The starting weights $w_{n,k}$ in (13) are to compensate for the reduced order of convergence.

However, computing the starting weights poses many difficulties in practice. Since the starting weights do not affect the stability or convergence of the solution, we do not include them in the computation and analysis in the subsequent sections. For some developments on the starting weights, we refer to [20, 26, 27].

2.3 Stability

For the analysis of the stability of an FLMM, consider the linear test problem

$$D_{t_0}^{\beta} y(t) = \lambda y(t), \quad y(t_0) = y_0, \qquad \lambda \in \mathbb{C}, \qquad 0 < \beta < 1 \tag{14}$$

for which the analytical solution is $y(t) = E_{\beta}(\lambda t^{\beta}) y_0$, where $E_{\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)}$ is the Mittag-Leffler function. For analytical stability, the solution $y(t)$ of the test problem (14) is stable in the sense that the series of $y(t)$ converges in the region

$$\Sigma_{\beta} = \{\xi \in C : |arg(\xi)| > \beta\pi/2\}.$$

The unstable region is then the infinite wedge with angle $\beta\pi$. (See Figure 1).

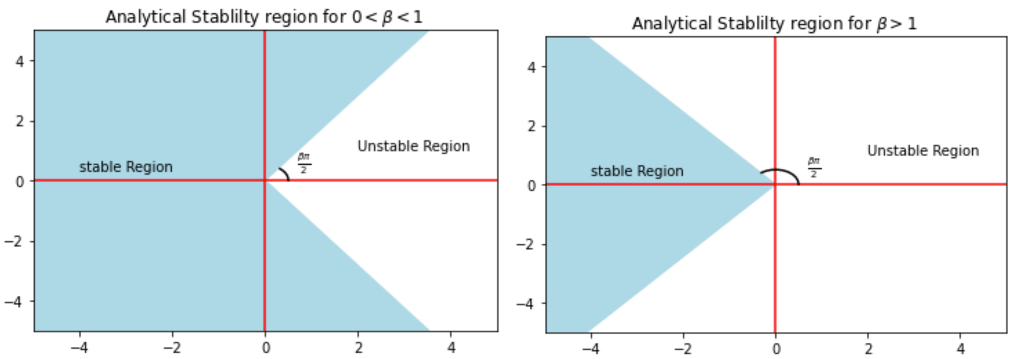

Figure 1. Analytical Stability regions.

For the numerical stability of FLMM, we have the following criteria:

Definition 2.3 *Let S be the numerical stability region of an FLMM. For an angle α, measured from the negative real axis, define the wedge $S(\alpha) = \{z : |arg(z) - \pi| \leq \alpha\}$. The FLMM is said to be*

1. $A(\alpha)$-stable if $S(\alpha) \subseteq S$
2. A-stable if it is $A(\pi - \beta\pi/2)$ stable.That is, $\Sigma_{\beta} \subseteq S$.
3. $A(\pi/2)$-stable when the entire left half of the complex plane is included in S.
4. Unconditionally stable if it is $A(0)$-stable. That is, the negative real line is included in S.

The stability region of an FLMM is also characterized by its generating function:

Theorem 2.4 [5] *The stability region of an FLMM with generating function $w(\xi)$ is given by*

$$S = \{W(\xi) : |\xi| > 1\} = \mathbb{C} \setminus S^c,$$

where $S^c = \{W(\xi): |\xi| \leq 1\}$ is the unstable region.

3. New FLMMs of order 4

In this section, we give the construction of these methods. Denote by $C^n(\mathbb{R})$ the class of functions having continuous n[th] derivatives.

3.1 Necessary Approximations

We need the following preparations.

Lemma 3.1 *Let $y(t) \in C^4(\mathbb{R})$ and $h > 0$. For $\mu \in \mathbb{R}$, $y(t + \mu h)$ can be interpolated with order 4 as*

$$y(t + \mu h) = p_0 y(t) + p_1 y(t - h) + p_2 y(t - 2h) + p_3 y(t - 3h) + O(h^4), \quad (15)$$

where $p_i, i = 0,1,2,3$ are the coefficients of the Lagrange interpolation polynomial approximation for a function of μ at the points $\mu_i = 0, -1, -2, -3$ given by

$$p_0 = \frac{(\mu+1)(\mu+2)(\mu+3)}{6},\ p_1 = -\frac{\mu(\mu+2)(\mu+3)}{2},\ p_2 = \frac{\mu(\mu+1)(\mu+3)}{2},\ p_3 = -\frac{\mu(\mu+1)(\mu+2)}{6}. \quad (16)$$

Proof. Consider $y(t + \mu h)$ as a function of μh interpolated at points $\mu_i h = 0, -h, -2h$ and $-3h$. Then the coefficients p_i are the Lagrange interpolation basis functions $L_i(\mu h) = \prod_{j=0,j\neq i}^{3} \frac{\mu h - \mu_j h}{\mu_i h - \mu_j h}$ which reduce to $p_i, i = 0,1,2,3$. The error of the interpolation is given by $E = \frac{1}{4!}\prod_{j=0}^{3} (\mu h - jh) y^{(4)}(t + \xi h) = O(h^4)$, where $\xi \in (-3,0)$. ■

Lemma 3.2 *The second derivative of a function $f(x)$ can be approximated by the backward difference forms of order 2 as*

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2}(2f(x) - 5f(x - h) + 4f(x - 2h) - f(x - 3h)) + O(h^2) \quad (17)$$

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2}(3f(x - h) - 8f(x - 2h) + 7f(x - 3h) - 2f(x - 4h)) + O(h^2). \quad (18)$$

Proof. Taylor series expansions. ■

Lemma 3.3 *The shifted GL approximation* (7) *with shift $r = \frac{\beta}{2}$ has order 2 super convergence in* (10) *with the coefficients of the odd order terms $a_{2k+1} = 0$ for $k = 0,1,2,\cdots$. Moreover, we have $a_2 = \frac{\beta}{24}$.*

Proof. The generating function of the GL approximations in (6) and (7) is $W(z) = (1 - z)^\beta$.
The function $G(x) = \frac{1}{x^\beta}(1 - e^{-x})^\beta e^{\frac{\beta}{2}x}$ in (9) is an even function since

$$G(-x) = \frac{1}{(-x)^\beta}(1 - e^{x})^\beta e^{-\frac{\beta}{2}x} = \frac{1}{(-x)^\beta}(-1)^\beta e^{\beta x}\,(1 - e^{-x})^\beta e^{-\beta x} e^{\frac{\beta}{2}x} = G(x).$$

Hence, the odd order terms of the series expansion of $G(x)$ are zero. Moreover, expanding for the first few terms reveals $a_0 = 1, a_2 = \frac{\beta}{24}$. ■

From Theorem 2.2 again, we have from (10),

$$\delta^{\beta}_{\tau,\frac{\beta}{2}} y(t_n) = \frac{1}{\tau^\beta}\sum_{k=0}^{\infty} g_k^{(\beta)} y\left(t_n - \left(k - \frac{\beta}{2}\right)\tau\right) = D^{\beta}_{t_0} y(t_n) + a_2\tau^2 D^{\beta+2}_{t_0} y(t_n) + (\tau^4). \quad (19)$$

Writing $D^{\beta+2}_{t_0} = D^2 D^{\beta}_{t_0}$, we replace the fractional derivative $D^{\beta}_{t_0} y(t_n)$ in (19) by $f(t_n, y(t_n))$:

$$\sum_{k=0}^{\infty} g_k^{(\beta)} y\left(t_{n-k} + \frac{\beta\tau}{2}\right) = \tau^\beta f(t_n, y(t_n)) + a_2\tau^{2+\beta} D^2 f(t_n, y(t_n)) + O(\tau^{4+\beta}). \quad (20)$$

We approximate $y(t_{n-k} + \beta\tau/2)$ by (15) in Lemma 3.1 with $\mu = \beta/2$ and $t = t_{n-k}$. Moreover, approximate the second derivative in (20) by (17) in Lemma 3.2. Then, we have from (16) with

$$p_0 = \frac{(\beta+2)(\beta+4)(\beta+6)}{48},\ p_1 = -\frac{\beta(\beta+4)(\beta+6)}{16},\ p_2 = \frac{\beta(\beta+2)(\beta+6)}{16},\ p_3 = -\frac{\beta(\beta+2)(\beta+4)}{48}. \quad (21)$$

$$\begin{aligned}\sum_{k=0}^{\infty} g_k^{(\beta)}[p_0 y(t_{n-k}) + p_1 y(t_{n-k-1}) + p_2 y(t_{n-k-2}) + p_3 y(t_{n-k-3})]\\ = \tau^\beta[f(t_n, y(t_n)) + a_2(2f(t_n, y(t_n)) - 5f(t_{n-1}, y(t_{n-1})) + 4f(t_{n-2}, y(t_{n-2}))\\ - f(t_{n-3}, y(t_{n-3})))] + O(\tau^{4+\beta}).\end{aligned} \quad (22)$$

Dropping the error term, with the notations in (12), equation (22) gives an implicit FLMM scheme

$$\sum_{k=0}^{\infty} g_k^{(\beta)}(p_0 y_{n-k} + p_1 y_{n-1-k} + p_2 y_{n-2-k} + p_3 y_{n-3-k}) = \tau^\beta[f_n + a_2(2f_n - 5f_{n-1} + 4f_{n-2} - f_{n-3})]. \quad (23)$$

Again, approximating the second derivative in (20) by (18) in Lemma 3.2, with the same operations and notations, we get another implicit FLMM:

$$\sum_{k=0}^{\infty} g_k^{(\beta)}(p_0 y_{n-k} + p_1 y_{n-1-k} + p_2 y_{n-2-k} + p_3 y_{n-3-k}) = \tau^\beta[f_n + a_2(3f_{n-1} - 8f_{n-2} + 7f_{n-3} - 2f_{n-4})] \quad (24)$$

For brevity of presentation, we call these FLMMs in (23) and (24) as NFLMM4.1 and NFLMM4.2 respectively.

3.2 Implementation

We use the following notations:
For a sequence $a = \{a_k\}$, we denote the finite vector $[a_i, a_{i+1}, \dots, a_j]$ as $a_{i:j}$. For given two sequences a, b, the convolution of two equal sized vectors $a_{i:j}, b_{l:m}$, , with $j - i + 1 = m - l + 1$, as $(a_{i:j} * b_{l:m}) = \sum_{k=i}^{j} a_k b_{n-k}$, where $n = m + i = l + j$.
Using these notations, the two proposed NFLMMs can be written as

$$(w_{0:n} * y_{0:n}) = \tau^\beta (q_{0:m} * f_{n-m:n}), \quad (25)$$

where $y = \{y_k\}$, $f = \{f_k\}$, $g = \{g_k\}$ and $w = \{w_k\}$ with w_k given by

$$w_k = p_0 g_k + p_1 g_{k-1} + p_2 g_{k-2} + p_3 g_{k-3} = p * g_{k-3:k}, \quad k = 0,1, \dots,$$

where we have assumed $g_{-1} = g_{-2} = g_{-3} = 0$. The coefficient vectors p, q are given by

$$p = [p_0, p_1, p_2, p_3] \text{ with } \mu = \beta/2, \quad (26)$$

for NFLMM4.1: $q = [1 + 2a_2, -5a_2, 4a_2, -a_2], \; m = 3,$ (27)

for NFLMM4.2: $q = [1, 3a_2, -8a_2, 7a_2, -2a_2], \; m = 4.$ (28)

Extracting terms of y_n, we write (25) as

$$w_0 y_n + (w_{1:n} * y_{0:n-1}) = q_0 \tau^\beta f_n + \tau^\beta (q_{1:m} * f_{n-m:n-1}). \quad (29)$$

First, we consider linear FIVP with $f(t, y) = \lambda y(t) + F(t)$. Then, equation (29) becomes, after solving for y_n,

$$y_n = \frac{1}{w_0 - q_0 \lambda \tau^\beta}[q_0 \tau^\beta F_n - s_{n-1}],$$

where $F_n = F(t_n)$ and $s_{n-1} = (w_{1:n} * y_{0:n-1}) - \tau^\beta(q_{1:m} * f_{n-m:n-1})$ which is independent of y_n.

Algorithm 1: Linear FIVP Solver

1. Input β, τ, y_0, function $f(t)$
2. Compute $g = \{g_k\}$, using $g_0 = 1, g_k = (1 - \frac{\beta+1}{k}) g_{k-1}, k = 1,2, \dots, N$.
3. Compute the convolution $w = p * g = [(p * g_{k-m:k}): k = 0,1, \dots, N]$.
4. For $n = 1,2, \dots, N$,
5. $s_{n-1} = (w_{1:n} * y_{0:n-1}) - \tau^\beta (q_{1:m} * f_{n-m:n-1})$
6. $y_n = \frac{1}{w_0 - q_0 \lambda \tau^\beta}[q_0 \tau^\beta F_n - s_{n-1}]$.

Next, for the nonlinear FIVP, we write equation (29) in the unknown $y = y_n$:

$$H(t_n, y_n) + s_{n-1} = 0, \quad (30)$$

where $H(t, y) = w_0 y - q_0 \tau^\beta f(t, y)$.
We use Newton-Raphson iteration to solve (30) for y_n using the initial seed $y_{n,0} = y_{n-1}$.
The derivative of $H(t, y)$ with respect to y is $H_y(t, y) = w_0 - q_0 \tau^\beta f_y(t, y)$,
where $f_y = \frac{\partial}{\partial y} f(t, y)$. Then, the algorithm for non-linear FIVP is given by

Algorithm 2: Non-linear FIVP Solver

1. Input β, τ, y_0, functions $f(t, y), f_y(t, y)$
2. compute $g = \{g_k\}$, using $g_0 = 1, g_k = (1 - \frac{\beta+1}{k}) g_{k-1}, k = 1,2, \dots, N$.
3. Compute $w = p * g = [p * g_{k-3:k}: k = 0,1, \dots, N]$.
4. Define functions $H(t, y) = w_0 y - q_0 \tau^\beta f(t, y), H_y(t, y) = w_0 - q_0 \tau^\beta f_y(t, y)$
5. For $n = 1,2, \dots, N$,
6. $s_{n-1} = (w_{1:n} * y_{0:n-1}) - \tau^\beta (q_{1:m} * f_{n-m:n-1})$
7. $y_n = Newton(H(t_n, y) + s_n, H_y(t_n, y), y_{n-1})$

where the function $Newton(h(y), h'(y), y^{(0)})$ performs the Newton-Raphson iterations to compute the root of $h(y) = 0$ with initial seed $y^{(0)}$.

4. Analysis of linear stability

The generating functions of the new implicit schemes NFLMM4.1 and NFLMM4.2 are given by:

$$W_{4.1}(\xi;\beta)=\frac{(1-\xi)^{\beta}P(\xi)}{Q_{4.1}(\xi)} \quad \text{and} \quad W_{4.2}(\xi;\beta)=\frac{(1-\xi)^{\beta}P(\xi)}{Q_{4.2}(\xi)} \tag{31}$$

respectively, where $P(\xi), Q_{4.1}(\xi)$ and $Q_{4.2}(\xi)$ are polynomials for which the coefficients are given by (26),(27) and (28) repectively. The following elementary results on complex numbers is useful.

Lemma 4.1 *Let z, w, be two distinct non-zero complex numbers. Then, the following are equivalent.*

(i) zw is a real.

(ii) z/w is real.

(iii) Both z,w are real or z is a real multiple of $\overline{w}$.

Proof. $(i) \Leftrightarrow (ii)$ and $(i) \Leftarrow (iii)$ are obvious. It is enough to prove $(i) \Rightarrow (iii)$.
Suppose that $z=x+iy, w=a+ib$. Then $zw=xa-yb+i(xb+ya)$ is real implies $xb+ya=0$. Then, for a real α, $x/a=-y/b=\alpha\neq 0$ which gives $x=\alpha a$ and $y=-\alpha b$. Hence $z=\alpha\overline{w}$. ■

In order to analyze the stability properties of the methods, we consider the unstable regions $\{W_{4.x}(\xi;\beta):|\xi|\leq 1\}$, $x=1,2$ and their properties.

Theorem 4.2 *The generating functions $W_{4.x}(\xi;\beta), x=1,2$, in* (31) *have the following properties:*

1. *ξ is real if and only if $W_{4.x}(\xi;\beta)$ is real. Moreover, $\xi\in[-1,1]$ if and only if $0\leq W_{4.x}(\xi;\beta)\leq W_{4.x}(-1;\beta)$.*
2. *If $\parallel \xi \parallel=1$ and $\Im(\xi)>0(<0)$, then $\Im(W_{4.x}(\xi;\beta)<0(>0)$.*

Proof. It is enough to prove the sufficiency of both statements.

1. Thanks to Lemma 4.1 that $W_{4.x}(\xi,\beta)$ is real if and only if the factors $(1-\xi)^{\beta}, P(\xi)$ and $Q_{4.x}(\xi)$ are all real. Hence, ξ is real.
2. If $-1\leq\xi\leq 1$, we have $0\leq 1-\xi\leq 2$. Clearly, then, $(1-\xi)^{\beta}$ is decreasing.
   Writing $P(\xi)$, $Q_{4.1}(\xi)$ and $Q_{4.2}(\xi)$ as polynomial of $(1-\xi)$, we get

$$P(\xi)=1+\frac{\beta}{2}(1-\xi)+\frac{\beta(\beta+2)}{8}(1-\xi)^2+\frac{\beta(\beta+2)(\beta+4)}{48}(1-\xi)^3,$$
$$Q_{4.1}(\xi)=1+a_2(1-\xi)^2+a_2(1-\xi)^3,$$
$$Q_{4.2}(\xi)=1+a_2(1-\xi)^2+a_2(1-\xi)^3-2a_2(1-\xi)^4,$$

Noticing that $\frac{\beta(\beta+2)(\beta+4)}{48}>a_2$ for NFLMM4.1, we immediately see that $\frac{P(\xi)}{Q_{4.1}(\xi)}$ is also decreasing. Hence, $W_{4.1}(\xi;\beta)$ is decreasing and $W_{4.1}(1;\beta)=0\leq W_{4.1}(\xi;\beta)\leq W_{4.1}(-1;\beta)$.
As for NFLMM4.2, $Q_{4.2}(\xi)$ is of degree 4 having an additional term with a negative coefficient. Hence, it is increasing. Therefore, $W_{4.2}(\xi;\beta)$ is decreasing and thus, $W_{4.2}(1;\beta)=0\leq W_{4.2}(\xi;\beta)\leq W_{4.2}(-1;\beta)$. ■

Theorem 4.3 *The unstable regions of NFLMM4.1 and NFLMM4.2 are bounded and symmetric about the real axis for* $0<\beta\leq 1$.

Proof. For the boundedness, we see that the numerator part of $W(\xi;\beta)$, with $|\xi|\leq 1$,

$$|(1-\xi)^{\beta}P(\xi)|\leq(1+|\xi|)^{\beta}\left(|p_0|+|p_1||\xi|+|p_2||\xi|^2+|p_3||\xi|^3\right)\leq 2^{\beta}(p_0-p_1+p_2-p_3)=2^{\beta}P(-1),$$

where we have used the facts that $p_0,p_2>0$ and $p_1,p_3<0$.
For the denominator part for NFLMM4.1,

$$|Q_{4.1}(\xi)|=|1+a_2(2-5\xi+4\xi^2-\xi^3)|\geq|1+2a_2-a_2(5+4+1))|$$
$$=1-\frac{\beta}{3}>0 \text{ for } 0<\beta<3.$$

For NFLMM4.2,

$$|Q_{4.2}(\xi)|=|1+a_2(3\xi-8\xi^2+7\xi^3-2\xi^4)|\geq|1-a_2(3+8+7+2)|$$
$$=Q(-1)=1-\frac{5\beta}{6}>0 \text{ for } 0<\beta<6/5.$$

Hence, for $0<\beta\leq 1$ and for $|\xi|\leq 1$,

$$|W_{4.x}(\xi;\beta)|=\frac{|1-\xi|^{\beta}|P(\xi)|}{|Q_{4.x}(\xi)|}<\infty, \quad \text{x = 1,2.}$$

Since $W_{4.x}(\overline{\xi};\beta)=\overline{W_{4.x}(\xi;\beta)}$, we immediately see that the unstable regions are symmetry about the real axis. ■
Since the NFLMM4.x are of order 4, the Dahlquist barrier for FIVPs tells us that they are not A-stable [13] .

Therefore, we look for the $A\left(\frac{\pi}{2}\right)$-stability of these methods, that is, if the methods are stable in the entire left complex plane.

Theorem 4.4 There are threshold values $\beta^*_{4.x} \in (0,1)$ such that the methods NFLMM4.x, $x = 1,2$, are $A\left(\frac{\pi}{2}\right)$-stable for $0 < \beta < \beta^*_{4.x}$.

Proof: First note that the genrations functions $W_{4.x}(\xi;\beta)$ are continuous function in β, where ξ is fixed.
Now, for $|\xi| \le 1$, from (31) with (21), we have $W_{4.1}(\xi, 0_+) = 1 > 0$. Hence there exists a neighbohood $(0,\epsilon)$ for β such that $W_{4.1}(\xi,\beta) > 0$ for all $|\xi| \le 1$.
Again, for $\xi_i = i$ and $\beta = 1$, we have

$$W_{4.1}(\xi_i, 1) = \frac{(1-i)(p_0 + ip_1 - p_2 + ip_3)}{1 + \frac{1}{24}(2 - 5i - 4 + i)} = -0.528 - 3.096i.$$

So, $\Re\big(W_{4.1}(\xi_i, 1)\big) = -0.528 < 0$. Hence, there is a neighborhood $(1-\epsilon, 1)$ for β such that $W_{4.1}(\xi_i,\beta) < 0$. Let $\beta^*_{4.1} = \max\{\beta : W_{4.1}(\xi,\beta) \ge 0\}$. Then $0 < \beta^*_{4.1} < 1$.

The proof for $W_{4.2}(\xi,\beta)$ is analogooius with $W_{4.1}(\xi_i, 1) = -0.1927 - 3.358i$. ■

Numerical computation by interval bisection shows that $\beta^*_{4.1} = 0.82960$ and $\beta^*_{4.2} = 0.85024912$ for the NFLMM4.1 and NFLMM4.2 respectively.

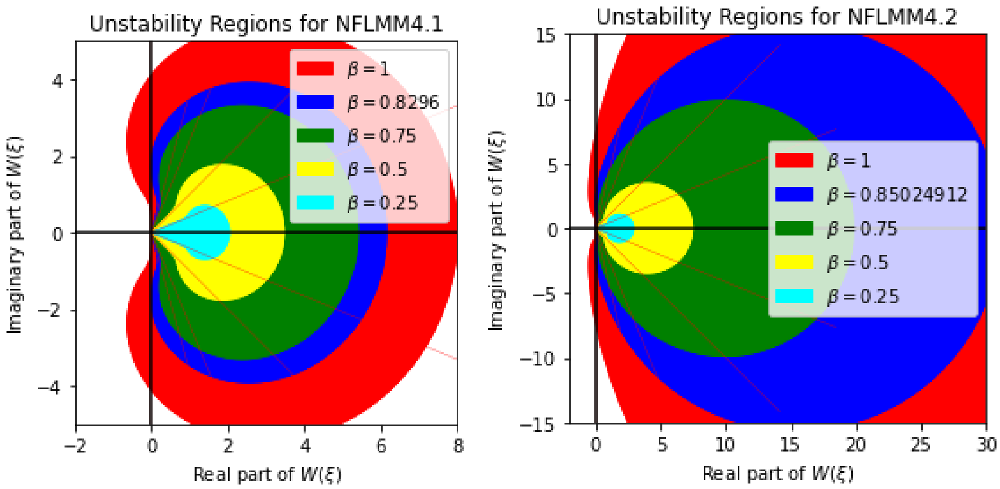

Figure 2. Stability regions of NFLMM4.x for some β values.

In Figure 2, the unstable regions of the two methods NFLMM4.x, x=1,2 are shown for different fractional orders $0 < \beta \le 1$. The straight lines in the figures represent the stability region boundaries of the methods, where the stability regions are shown on the left side of the lines. These lines also correlate to the analytical stability region's boundary Σ_β. It is clear from the figure that the unstable regions surpass the A-stable boundaries for all values of β. Thus, the methods are verified to be not A-stable. The regions in blue are the unstable regions for the threshold values $\beta^*_{4,x}$ which indicate $A\left(\frac{\pi}{2}\right)$-stable boundaries.

5. Comparisons

In this section, we compare the order 4 NFLMM4.x with FAM3 and FBDF4 for their performances in terms of computations and stability.

5.1 Numerical comparisons

Consider the linear FIVP used in [9].

$$D_t^\beta y(t) = f\big(t, y(t)\big) = \lambda y(t) + F(t), \;\; 0 \le t \le 1, \;\; 0 < \beta \le 1,$$

with the initial condition $y(0) = 0$,
where $F(t) = \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} t^{n-\beta} - \frac{\Gamma(n)}{\Gamma(n-\beta)} t^{n-1-\beta} + t^n - t^{n-1}$ with $\lambda = -1$.
The exact solution is given by: $y(t) = t^n - t^{n-1}$, where $n = 5$.
The linear equation was solved by using the schems FAM3, FBDF4, NFLMM4.1 and NFLMM4.2 in (23) and (24) with different values of fractional orders $\beta = 0.4, 0.6$ and 0.8. The problem is computed on the domain $[0,1]$, with

$N_j = 2^j,\ j = 3,4,\ldots,11$ as the number of subintervals and $\tau_j = \frac{1}{N_j}$ as the step size. The maximum errors E_j for step size τ_j are compared for the methods. The order of the method NFLMM4.1 is computed by the formula

$$p_{j+1} = \frac{log(E_{j+1}/E_j)}{log(\tau_{j+1}/\tau_j)}.$$

The orders of other methods are nearly the same and are not presented.
The results obtained are listed in Table 1, 2 and 3 respectively.

Table 1. Comparing Maximum errors for $\beta = 0.4$.

N_j	NFLMM4.1	NFLMM4.2	FBDF4	FAM3	Order
8	5.968e-04	2.702e-04	8.327e-04	2.641e-04	-
16	4.171e-05	1.752e-05	5.952e-05	1.785e-05	3.83859
32	2.741e-06	1.115e-06	3.947e-06	1.157e-06	3.92775
64	1.754e-07	7.033e-08	2.537e-07	7.361e-08	3.96556
128	1.109e-08	4.415e-09	1.607e-08	4.641e-09	3.98317
256	6.974e-10	2.766e-10	1.011e-09	2.913e-10	3.99167
512	4.371e-11	1.730e-11	6.341e-11	1.824e-11	3.99585
1024	2.736e-12	1.082e-12	3.972e-12	1.141e-12	3.99789
2048	1.711e-13	6.772e-14	2.523e-13	7.147e-14	3.99843

Table 2. Comparing Maximum errors for $\beta = 0.6$.

N_j	NFLMM4.1	NFLMM4.2	FBDF4	FAM3	Order
8	1.161e-04	6.265e-04	1.361e-03	3.429e-04	-
16	8.129e-05	4.207e-05	9.629e-05	2.288e-05	3.83713
32	5.334e-06	2.715e-06	6.352e-06	1.474e-06	3.92967
64	3.411e-07	1.723e-07	4.072e-07	9.346e-08	3.96670
128	2.156e-08	1.085e-08	2.577e-08	5.884e-09	3.98378
256	1.355e-09	6.809e-10	1.620e-09	3.691e-10	3.99199
512	8.494e-11	4.264e-11	1.016e-10	2.311e-11	3.99602
1024	5.316e-12	2.667e-12	6.370e-12	1.445e-12	3.99812
2048	3.323e-13	1.663e-13	4.168e-13	9.028e-14	3.99975

Table 3. Comparing Maximum errors for $\beta = 0.8$.

N_j	NFLMM4.1	NFLMM4.2	FBDF4	FAM3	Order
8	1.982e-04	1.202e-04	1.972e-03	3.857e-04	-
16	1.386e-05	8.215e-05	1.385e-04	2.546e-05	3.8374
32	9.093e-06	5.336e-06	9.102e-06	1.632e-06	3.93067
64	5.812e-07	3.397e-07	5.823e-07	1.032e-07	3.96753
128	3.672e-08	2.142e-08	3.681e-08	6.492e-09	3.98425
256	2.307e-09	1.345e-09	2.314e-09	4.070e-10	3.99222
512	1.446e-10	8.423e-11	1.450e-10	2.548e-11	3.99611
1024	9.053e-12	5.272e-12	9.094e-12	1.595e-12	3.99776
2048	5.701e-13	3.325e-13	6.024e-13	9.947e-14	3.98928

Since all the four methods are of order 4, the computational solutions for all choices of discretization are expected to be nearly the same.

As for the computational cost, the weights of the NFLMM4.x need only a linear combination of the Grunwald coefficients $g_k^{(\beta)}$, that have the simplest computational cost. The weights of FBDF4 obviously require computations using Miller's formula with four prior weights.

As for the memory requirement, the FAM3 requires keeping all the f_n values stored during the iteration to be used on the right side of scheme (25). The NFLMM4.x require only the last three or four values of f_n as in (23) and (24).

5.2 Comparison of stability

We compare the stability regions of the four implicit FLMMs. The generating functions FBDF4 and FAM3 are provided below:

$$W_{FBDF4}(\xi) = \left(\frac{25}{12} - 4\xi + 3\xi^2 - \frac{4}{3}\xi^3 + \frac{1}{4}\xi^4\right)^\beta \text{ and } W_{FAM3}(\xi) = \frac{(1-\xi)^\beta}{q_0+q_1\xi+q_2\xi^2+q_3\xi^3},$$

where $q_i, i = 0,1,2,3$, are given by

$$q_0 = 1 - \frac{5}{6}\beta + \frac{11}{48}\beta^2 - \frac{1}{48}\beta^3, \qquad q_1 = \frac{31}{24}\beta - \frac{9}{16}\beta^2 + \frac{1}{16}\beta^3,$$
$$q_2 = -\frac{7}{12}\beta + \frac{7}{16}\beta^2 - \frac{1}{16}\beta^3, \qquad q_3 = \frac{1}{8}\beta - \frac{5}{48}\beta^2 + \frac{1}{48}\beta^3.$$

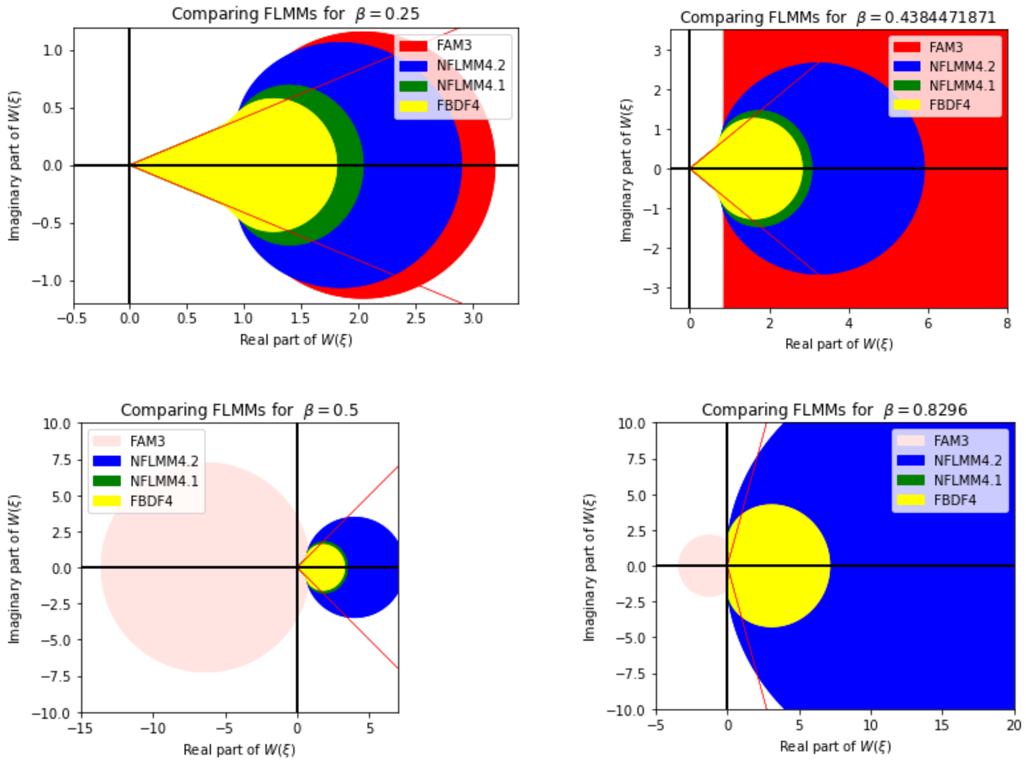

Figure 3. Comparing stability regions for FLMMs of order 4.

Since the FLMM methods with orders greater than 2 are not A-stable, the $A\left(\frac{\pi}{2}\right)$-stable and $A(0)$-stable could be used as comparison tools of those methods.

As shown in Figure 3, the unstable region for many values of β is on the right side of the complex plane. However, there are some β values for which the unstable region also extends to the left side, such as $\beta = 1$. The intervals for β where the FLMMs are $A\left(\frac{\pi}{2}\right)$-stable were calculated numerically. The β^* values for which the intervals $0 < \beta \leq \beta^*$ gives the $A(\frac{\pi}{2})$-stability are given in Table 4.

Table 4. Threshold β^* for $A\left(\frac{\pi}{2}\right)$-stability.

FAM3	NFLMM4.1	FBDF4	NFLMM4.2
0.4384471	0.82960	0.843895	0.85024912

For the $A\left(\frac{\pi}{2}\right)$-stability the NFLMM4.2 has the highest interval for β followed by FBDF4, NFLMM4.1 and FAM3. The lower interval size is gained for FAM3. Also note that as β approaches the $A\left(\frac{\pi}{2}\right)$-stability bound β^*, FAM3's A(0)-stability vanishes. For $\beta > \beta^*$ the stability region becomes bounded and falls on the left complex plane, resulting in only conditional stability.

7. Conclusion

We analyzed and compared the new fractional linear multistep methods NFLMM4.1 and NFLMM4.2 with FAM3 and FBDF4 methods of order 4. We see that NFLMM4.2 is $A\left(\frac{\pi}{2}\right)$-stable over a wider fractional-order interval while FAM3 displays $A\left(\frac{\pi}{2}\right)$-stablity for a small range of β values. Furthermore, the proposed methods have lower computing cost and minimal storage compared to FAM3.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgment

The authors are greatful to the reviewers for their contructive comments.

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Received 25 March 2023
Accepted 30 May 2023